

A new probabilistic approach to non local and fully non linear second order partial differential equations

Jocelyne Bion-Nadal
 CMAP (UMR CNRS 7641)
 Ecole Polytechnique F-91128 Palaiseau cedex
 jocelyne.bion-nadal@cmap.polytechnique.fr

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Abstract

We prove that weakly continuous solutions to martingale problems admit a canonical regular conditional probability distribution. This allows for the construction of time consistent convex dynamic procedures in a non dominated setting. Making use of the martingale problem approach for continuous diffusions and diffusions with Levy generator, we give an explicit construction of such procedures having furthermore a Feller property.

These procedures lead to viscosity solution of fully non linear second order partial differential equations in case of continuous diffusions. In case of diffusions with Levy generator this provides a probabilistic approach for the resolution of non local fully non linear second order PDE.

Keywords: Time consistency, convex duality, Feller processes, Viscosity solutions to second order PDE

MSC:

1 Introduction

Starting from the solution of the non linear heat equation, Peng [27, 28] has introduced the notion of G -expectation $(\mathcal{E}_t^G)_{t \in \mathbb{R}_+}$ which is a sublinear time consistent dynamic procedure defined on continuous functions on Ω . The state space Ω is there equal to the set of continuous paths, and $\mathcal{E}_t^G(X)$ is defined by a stepwise evaluation of the partial differential equation (PDE). Denis et al [12] have then proved that $\mathcal{E}_0^G(X) = \sup_{P^\theta \in \mathcal{P}} E_{P^\theta}(X)$, where \mathcal{P} is a weakly compact set of probability measures which are non dominated

(i.e. there is no probability measure P such that $Q \ll P$ for all Q in \mathcal{P}). The properties of $(\mathcal{E}_t^G)_{t \in \mathbb{R}_+}$ are up to a minus sign the properties of a sublinear time consistent dynamic risk measure. Since the seminal papers of Peng on G -expectation, a new challenge is to develop a theory of time consistent dynamic procedures in a non dominated framework, and also to provide viscosity solutions to fully non linear second order partial differential equations.

Some recent works study the properties of these procedures, in the static case [6] and in the dynamic case [25, 7]. Other works construct time consistent dynamic procedures. The first construction of G -expectation by Peng [27, 28] was made on the state of continuous paths from the non linear heat equation $-\partial_t v(t, x) - G(D^2 v(t, x)) = 0$, $G(A) = \sup_{\gamma \in \Theta} \text{Tr}(\gamma \gamma^* A)$ (Θ bounded) with boundary condition $v(0, x) = f(x)$. Starting from the unique viscosity solution of a second order PDE containing a non local term, this construction has been extended in [18] to produce a time consistent sublinear procedure on the set of càdlàg paths. In both cases G is independent of t and x and the G -heat equation corresponds to diffusions with no drift and a diffusion coefficient varying between two bounds. These works of Peng *et al.* have motivated other works constructing time consistent dynamic procedures on the set of continuous paths, in a non dominated framework. Nutz has produced time consistent sublinear procedures allowing the G coefficient to vary with both t and x (or equivalently with ω in the non Markovian case) in [23], or allowing both the diffusion coefficient and the drift term to vary between bounds [24]. In both cases the construction is made ω by ω . Starting from a given set of probability measures for all t and ω , satisfying some compatibility conditions and analytic properties, Nutz and van Handel [26] construct also sublinear time consistent procedures on the space of continuous paths. Soner et al [30, 31] have constructed convex (and not only sublinear) time consistent dynamic procedures in the setting of diffusions as well. Their approach makes use of a solution of a Backward Stochastic Differential Equation (BSDE) associated to every probability measure in the set \mathcal{P} and then solves an “aggregation problem”. The diffusion coefficient varies in a domain independent of t and x . In [31] it is proved that this construction gives rise to viscosity solutions to fully non linear second order PDE.

Time consistent dynamic procedures on a filtered probability space or up to a minus sign dynamic risk measures have been studied in many papers [3, 4, 8, 10, 20]. Risk measures are characterized by their dual representation. In particular, in the static case [15, 16], $\rho_{0,t}(X) = \sup_{Q \in \mathcal{P}} (E_Q(-X) - \alpha(Q))$ where \mathcal{P} is a set of probability measures all absolutely with respect to P .

Sublinear time consistent risk measures are fully characterized by a stable set of probability measures [10]. Convex time consistent dynamic risk measures are described by a stable set of probability measures \mathcal{P} and a local penalty defined on \mathcal{P} satisfying the cocycle condition [3, 4]. In the particular case of the Brownian filtration, time consistent dynamic risk measures are limits of solutions of BSDE [11].

In order to generalize the construction of time consistent dynamic procedures to the case of non dominated probability measures, the main point is to understand the notion of stable set of probability measures in this new framework. Indeed the usual notion of stable set involves Q -conditional expectation. However Q -conditional expectation is defined up to a Q -null set and this is a drawback when one considers non equivalent probability measures. On a Polish space Ω , for all sub σ - algebra \mathcal{B} of the Borel σ algebra, there exists always a measurable version of the Q -condition expectation called regular conditional probability distribution given \mathcal{B} ([36], Theorem 1.1.6.). However there is no unicity of the Q -regular conditional distribution given \mathcal{B} . In order to extend the notion of stability by bifurcation [3] to a set of non dominated probability measures as well as the local condition for a penalty [3], one needs to be able to make a coherent choice for a measurable version of the conditional expectation.

The first goal of the present paper is to prove that weakly continuous solutions to a martingale problem (Section 2) is a very nice setting in which there exists a canonical version of the conditional expectation (Section 3). This allows to extend the notion of stability to a set of probability measures which are non dominated (Section 4). This canonical version has furthermore continuity properties (Section 5). Specializing to the case of continuous diffusions or to the case of diffusions with Levy generator, we then construct penalties having a Feller property (Section 7), satisfying the local condition and the cocycle condition. The canonical version of conditional expectation joint with the Feller penalties give a generic and constructive method to produce \mathcal{T} -time consistent convex dynamic procedures on the space of continuous paths and also on the space of càdlàg paths in a non dominated framework. The set \mathcal{T} can be any subset of \mathbb{R}_+^* in particular it can be \mathbb{R}_+^* or a discrete subset of \mathbb{R}_+^* . This construction generalizes to the non dominated framework the construction of [5]. It can be used to construct a great variety of time consistent dynamic procedures in non dominated framework. In the case of continuous paths, the underlying set of probability measures \mathcal{P} is a “stable set” generated by probability measures solution to a martingale problem associated to a continuous diffusion with diffusion coefficient $a(t, x)$ and drift coefficient $b(t, x)$ continuous bounded, $a(t, x)$ invertible for

all (t, x) such that (a, b) takes values in a multivalued Borel set. In the case of càdlàg paths, the underlying set of probability measures \mathcal{P} is the “stable set” generated by probability measures solutions to a martingale problem associated to diffusions with Levy generator with coefficients $a(t, x)$ $b(t, x)$ satisfying the above conditions and the jump measure $M(t, x)$ satisfying the hypothesis of [32], (a, b, M) taking values in a multivalued Borel set. The procedure is defined on the closure (for the norm $\sup_{P \in \mathcal{P}} E_P(|X|)$) of the lattice vector space of continuous coordinate functions $f(X_{t_1}, \dots, X_{t_k})$ (f continuous on $(\mathbb{R}^n)^k$) and also on the cone $f(X_{t_1}, \dots, X_{t_k})$, where f is lower semi continuous bounded from below. A construction for a time consistent procedure on càdlàg paths is independently proposed in [19], but within a very limited and specific framework: the set of probability measures is generated by probability measures solutions to the martingale problem associated to diffusions with Levy generator where there is no drift, the diffusion coefficient and the jump measure depend only of t (not on x), there is no multivalued Borel mapping, and no penalty function. Furthermore the procedure is defined on the uniformly continuous functions on the space of càdlàg paths, which does not allow to study PDEs in the context of càdlàg paths.

Our last goal is to prove as in [5] that these procedures give a new probabilistic approach to second order PDE (Section 9). Indeed, in the case $\mathcal{T} = \mathbb{R}_+$, these procedures applied to the random variable $h(X_t)$ (h lower semi-continuous bounded from below) give rise to a time consistent convex Feller process $\tilde{h}(s, X_s)$. The function \tilde{h} is lower semi-continuous on $[0, t] \times \mathbb{R}^n$. Making use of the martingale property of the probability measures, we prove that this lower semi-continuous function is a viscosity supersolution of a second order PDE:

$$\begin{cases} -\partial_u v(u, x) - f(u, x, Dv(u, x), D^2v(u, x), \tilde{K}v(u, x)) &= 0 \\ v(t, x) &= h(x) \end{cases} \quad (1)$$

The non local term $\tilde{K}v(u, x)(y) = v(u, x + y) - v(u, x) - \frac{y^* \nabla v(u, x)}{1 + \|y\|^2}$ is specific of the case of diffusions with Levy generator. In case of continuous diffusions f depends only on $(u, x, Dv(u, x), D^2v(u, x))$, the PDE is a fully non linear second order partial differential equation. In this last case, our results can be compared to those of [31]. In [31], the function f can also depend on v but there are restrictive conditions on the dual of f which are not needed in our approach. This is due to the fact that the construction

in [31] relies on the existence of solutions to BSDE and also on the aggregation of these solutions. On the contrary in the present paper we have an explicit construction of the time consistent procedure. Furthermore the stability property of the set of probability measures and the conditions on the penalties imply the time consistency for the procedure (Section 6). As mentioned in [5], the time consistency for the procedure corresponds to the usual dynamic programming principle. Notice also that our probabilistic approach can be considered as a general control problem, where the set of control is a set of non dominated probability measures. As in [5], under some conditions the function \tilde{h} is continuous and is a viscosity solution of the PDE.

2 Martingale problem

We fix a finite horizon T . In the following for $r \leq T$, the state space Ω^r is either the set $\mathcal{C}([r, T], \mathbb{R}^n)$ of continuous paths on $[r, T]$ endowed with the topology of uniform convergence, or the set $\mathcal{D}([r, T])$ of càdlàg paths with the Skorokod topology [2]. Recall that Ω^r is a Polish space i.e. a complete separable metrizable space [2].

In both cases $(X_t)_{t \in \mathbb{R}^+}$ denotes the coordinate process, and \mathcal{B}_t^r is the σ -algebra generated by $\{X_u, r \leq u \leq t\}$.

2.1 Diffusions with continuous coefficients

As in [5] we consider the martingale problem introduced in [33]. In this subsection the state space is the set of continuous paths, $\Omega^r = \mathcal{C}^r = \mathcal{C}([r, T], \mathbb{R}^n)$. Let $a : [0, T] \times \mathbb{R}^n \rightarrow M_n(\mathbb{R})$ and $b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous bounded maps such that for all (t, x) , $a(t, x)$ is definite positive (i.e. a is strictly elliptic). For given $\theta \in \mathbb{R}^n$ let

$$Y_{r,t}^{a,b}(\theta) = \exp\{\theta^*(X(t) - X(r)) - \int_r^t \theta^* b(u, X(u)) du - \frac{1}{2} \int_r^t \theta^* a(u, X(u)) \theta du\} \quad (2)$$

Following [33] one says that the probability measure $Q_{r,y}^{a,b}$ on $(\Omega^r, \mathcal{B}^r)$ is a solution to the martingale problem (2) starting from y at time r if for all $\theta \in \mathbb{R}^n$, $(Y_{r,t}^{a,b}(\theta))_{t \geq r}$ is a martingale on $(\Omega^r, (\mathcal{B}^r)_t)_{t \geq r}, Q_{r,y}^{a,b}$, and if $Q_{r,y}^{a,b}(\{X_r = y\}) = 1$.

Denote $\mathcal{C}_c^\infty(\mathbb{R}^n)$ the set of \mathcal{C}^∞ functions with compact support. Denote $L_t^{a,b} = \frac{1}{2} \sum_1^n a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_1^n b_i(t, x) \frac{\partial}{\partial x_i}$. From [35] Theorem 2.1, the

above martingale problem is equivalent to

$$Z_{r,t}^{a,b} = f(X_t) - f(X_r) - \int_r^t L_u^{a,b}(f)(X_u) du \text{ is a } Q_{r,y}^{a,b}\text{-martingale for all } f \text{ in } \mathcal{C}_c^\infty(\mathbb{R}^n) \quad (3)$$

The study of the martingale problem has been extended in [32] to the case of diffusions with Levy generators.

2.2 Diffusions with Levy generators

In this subsection Ω^r is equal to $\mathcal{D}^r = \mathcal{D}([r, T], \mathbb{R}^n)$, the set of càdlàg paths endowed with the Skorokod topology [2].

Definition 2.1 *Hypothesis (M)*

A Borelian map M defined on $\mathbb{R}_+ \times \mathbb{R}^n$ with values in the set of σ -finite measures on $\mathbb{R}^n - \{0\}$ satisfies hypothesis (M) if for all Borelian subset Δ of $\mathbb{R}^n - \{0\}$,

$$\int_{\Delta} \frac{y}{1 + \|y\|^2} M(s, x, dy) \text{ is continuous bounded} \quad (4)$$

Let K_t^M be the operator associated to M :

$$K_t^M(f)(x) = \int [f(x+y) - f(x) - \frac{y^* \nabla f(x)}{1 + \|y\|^2}] M(t, x, dy) \quad (5)$$

Recall the following result from [32], Theorem 4.3.

Proposition 2.2 *For all a and b continuous bounded and a strictly elliptic, for all M satisfying the hypothesis (M), for all r in \mathbb{R}_+ and all y in \mathbb{R}^n , there is a unique probability measure $Q_{r,y}^{a,b,M}$ on $(\mathcal{D}^r, \mathcal{B}^r)$ solution to the martingale problem*

$$Z_{r,t}^{a,b,M} = f(X_t) - f(X_r) - \int_r^t (L_u^{a,b} + K_u^M)(f)(X_u) du \quad \forall f \in \mathcal{C}_c^\infty(\mathbb{R}^n) \quad (6)$$

starting from y at time r i. e. such that $Q_{r,y}^{a,b,M}(\{X_r = y\}) = 1$

2.3 Weakly continuous solution to a martingale problem

Motivated by the above martingale problem both for continuous diffusions and for diffusions with Levy generators, we consider in the following solutions to a martingale problem in a general setting. A finite horizon T is given. The state space can be either the set of continuous or of càdlàg paths.

Definition 2.3 *For all $s \in [0, T[$ and $y \in \mathbb{R}^n$, let $Q_{s,y}$ be the unique solution to the martingale problem Z on $(\Omega^s, \mathcal{B}_T^s)$, starting from y at time s . (Z means a whole family $(Z^i)_{i \in I}$.*

- $(Q_{s,y})_{y \in \mathbb{R}^n}$ is weakly continuous
if $Q_{s,y}$ is a continuous function of y for the weak topology.
- The martingale problem is additive if for all $i \in I$, for all $0 \leq r \leq s \leq t$, $Z_{r,s}^i$ is \mathcal{B}_s^r measurable, $Z_{r,s}^i$ is a right continuous function of s , and

$$Z_{r,t}^i = Z_{r,s}^i + Z_{s,t}^i \quad (7)$$

- The martingale problem is bounded if for all $0 \leq r \leq s$, $Z_{r,t}^i$ is bounded.

Definition 2.4 *Hypothesis H_Θ*

Θ is a set of Borelian maps defined on $\mathbb{R}_+ \times \mathbb{R}^n$ with values in a topological space E such that

1. for all $\theta \in \Theta$, for all $0 \leq r < t \leq T$ and for all y in \mathbb{R}^n , there is a unique probability measure $Q_{r,y}^\theta$ on $(\Omega_t^r, \mathcal{B}_t^r)$ solution to the additive bounded martingale problem Z^θ starting from y at time r .
2. for all $\theta \in \Theta$, and $r < T$, $(Q_{r,y}^\theta)_{y \in \mathbb{R}^n}$ is weakly continuous.

Continuous diffusions and diffusions with Levy generators provide examples for Θ :

Proposition 2.5 *The notations are those of Section 2.1. Let a and b be continuous bounded on $([0, T] \times \mathbb{R}^n)$ such that $a(s, x)$ is invertible for all (s, x) . Let $Z_{r,t}^{a,b}$ be given by equation (3). The unique solution $Q_{r,y}^{a,b}$ on $(\mathcal{C}^r, \mathcal{B}_T^r)$ to the additive martingale problem $Z_{r,t}^{a,b}$ starting from y at time r is weakly continuous.*

$$\Theta = \{(a, b) \text{ continuous bounded, } a \text{ invertible}\} \quad (8)$$

satisfies hypothesis H_Θ .

Proof The weak continuity follows Theorem 7.1 of [34]). See also Proposition 2.6 of [5] for a detailed argument. It follows easily from the definition that $Z_{r,t}^{a,b}$ is additive and bounded for all f in $\mathcal{C}_c^\infty(\mathbb{R}^n)$. \square
For diffusions with Levy generators we introduce now another hypothesis.

Definition 2.6 *Hypothesis M_c . The map M with values in σ finite measures on $\mathbb{R}^n - \{0\}$ satisfies hypothesis M_c if it satisfies hypothesis M and if*

$$\sup_{s,x} \int [||y||^2 1_{||y|| \leq 1} + ||y|| 1_{||y|| > 1}] M(s, x, dy) \leq C \quad (9)$$

The following Lemma is proved in [22] (cf the proof of Theorem 20).

Lemma 2.7 *Let A, B, C, D be strictly positive. For all $\epsilon > 0$ there is $K > 0$ and for all $\epsilon, \eta > 0$, there is $h > 0$ such that for all $0 \leq r \leq T$, for all a continuous strictly elliptic bounded by A for all b continuous bounded by B and M satisfying hypothesis M_c , and $||y|| \leq D$,*

$$Q_{r,y}^{a,b,M} [\sup_{r \leq v \leq T} ||X_v|| > K] \leq \epsilon \quad (10)$$

$$\forall u \in [r, T], \quad Q_{r,y}^{a,b,M} [\sup_{r \leq u \leq v \leq \inf(u+h, T)} ||X_v - X_u|| > \eta] \leq \epsilon \quad (11)$$

Notice that for given r and y , the above Lemma implies the relative weak compacity of the set of probability measures $Q_{r,y}^{a,b,M}$ satisfying the above conditions. The equation (11) is a “uniform right convergence”. This property is stronger than the property needed to prove the relative weak compacity (Theorem 13.2 of [2]). In particular it allows to prove the following proposition.

If $s_n > s$, the probability measure $Q_{s_n, y_n}^{a,b,M}$ on Ω^{s_n} can be identified with a probability measure on Ω^s still denoted $Q_{s_n, y_n}^{a,b,M}$ such that $Q_{s_n, y_n}^{a,b,M}(\{X_u = y_n, \forall s \leq u \leq s_n\}) = 1$.

Notation 2.8 *Let $0 \leq r \leq s \leq t \leq T$.*

- Denote $\pi_{[s,t]}^r$ the canonical projection of $\Omega^r = \Omega_T^r$ onto Ω_t^s .

$$\pi_{[s,t]}^r(\omega) = \omega|_{[s,t]} \quad (12)$$

- In case $r = s$ the projection $\pi_{[s,t]}^r$ will be denoted simply π_t^r

Proposition 2.9 *Let a be continuous strictly elliptic bounded by A , b continuous bounded by B and M satisfying hypothesis M_C . Assume that (s_n, y_n) has the limit (s, y) . Then*

1. - *If s_n is decreasing to s , with the above identification, $Q_{s_n, y_n}^{a, b, M}$ converges weakly to $Q_{s, y}^{a, b, M}$.*
- *If s_n is increasing to s , the image of $Q_{s_n, y_n}^{a, b, M}$ by $\pi_{[s, T]}^{s_n}$ converges weakly to $Q_{s, y}^{a, b, M}$.*
2. *For all (t_1, \dots, t_k) , $s \leq t_1 < \dots < t_k \leq T$, for all f continuous bounded on $(\mathbb{R}^n)^k$, $Q_{s_n, y_n}^{a, b, M}(f(X_{t_1}, \dots, X_{t_k}))$ has the limit $Q_{s, y}^{a, b, M}(f(X_{t_1}, \dots, X_{t_k}))$.*

Proof

From Lemma 2.7 and from Theorem 13.2 of [2]), it follows that $\{Q_{s_n, y_n}^{a, b, M}, n \in \mathbb{N}\}$ in case $s_n \geq s$, $(\{(Q_{s_n, y_n}^{a, b, M})(\pi_{[s, T]}^{s_n})^{-1}, n \in \mathbb{N}\})$ in case $s_n < s$ is relatively compact for the weak topology. The set of probability measures on $(\Omega^s, \mathcal{B}_T^s)$ is metrizable for the weak topology, thus there is a subsequence Q_k converging to Q for the weak topology on Ω^s . As in Section 13 of [2] denote T_Q the set of $t \in [s, T]$ for which the projection $\Pi_t : \omega \in \Omega^s \rightarrow \omega(t)$ is continuous except at points forming a Q -null set. It is proved in Section 13 of [2] that s, T belong to T_Q and that the complement of the set T_Q in $[s, T]$ is at most countable.

1. Step 1. We prove that equations (10) and (11) are also satisfied by Q for $r = s$. Notice that the weak convergence of Q_k to Q means that $Q_k(f)$ has the limit $Q(f)$ for continuous functions f but this convergence is not valid for general Borelian functions. Let T_Q^i be an increasing sequence of finite subsets of T_Q containing s and T such that $\cup_i T_Q^i$ is dense in $[s, T]$. It follows from the Mapping Theorem (Theorem 2.7. of [2]) and the inequality $R(G) \leq \liminf R_k(G)$ for all open set G and every sequence R_k weakly converging to R , that for all i ,

$$Q\left[\sup_{s \leq v \leq T, v \in T_Q^i} \|X_v\| > K\right] \leq \epsilon \quad (13)$$

$$\forall u \in [s, T] \cap T_Q^i, \quad Q\left[\sup_{u \leq v \leq \inf(u+h, T), v \in T_Q^i} \|X_v - X_u\| > \eta\right] \leq \epsilon \quad (14)$$

It follows from the monotone convergence Theorem that one can replace in the above equations T_Q^i by $\cup_i T_Q^i$. Equations (10) and (11) follow then for Q making use of the density of $\cup_i T_Q^i$ in $[s, T]$ and of the right continuity of X_v for all v .

2. step 2: We prove that for all (t_1, \dots, t_l) , $s \leq t_1 < \dots < t_l \leq T$, for all f continuous bounded on $(\mathbb{R}^n)^l$, $Q_k(f(X_{t_1}, \dots, X_{t_l}))$ has the limit $Q(f(X_{t_1}, \dots, X_{t_l}))$.

From [2] Section 13 this is true for all t_1, \dots, t_l , in T_Q , $s \leq t_1 < \dots < t_l \leq T$. The set T_Q is dense and contains s and T , thus for all $U = (t_1, \dots, t_l)$ there is a sequence $U^j = (t_1^j, \dots, t_l^j)$ of l -uplets T_Q valued decreasing to U . The function f being uniformly continuous on compact sets, it follows from equations (10) and (11) satisfied for all Q_k and Q , that for all n , for all $j \geq J$, all k , $Q_k(|f(X_{t_1}, \dots, X_{t_l}) - f(X_{t_1^j}, \dots, X_{t_l^j})|) < \epsilon$ and $Q(|f(X_{t_1}, \dots, X_{t_l}) - f(X_{t_1^j}, \dots, X_{t_l^j})|) < \epsilon$. From the convergence of $Q_k(f(X_{t_1^j}, \dots, X_{t_l^j}))$ to $Q(f(X_{t_1^j}, \dots, X_{t_l^j}))$, we then get the result.

3. Step 3. Let $s \leq t \leq T$. Let $Z_{s',t}^{a,b,M}$ be given by equation (6) for some function $f \in \mathcal{C}^\infty$ with compact support. It follows from hypothesis M_C that the function $(u, x) \in [0, T] \times \mathbb{R}^n \rightarrow L_u^{a,b}(f)(x) + K_u^M(f)(x)$ is continuous bounded. A similar argument as the above one proves that for all $\epsilon > 0$, there are $s \leq s_1 < \dots < s_p \leq T$ and a continuous bounded function $g_{s,t}$ on $(\mathbb{R}^n)^p$ such that for all Q and Q_k , $Q_k(|Z_{s,t}^{a,b,M} - g_{s,t}(X_{s_1} \dots X_{s_p})|) \leq \epsilon$, and $Q(|Z_{s,t}^{a,b,M} - g_{s,t}(X_{s_1} \dots X_{s_p})|) \leq \epsilon$. Making use of the martingale property of $(Z_{s,t}^{a,b,M})_{s \leq t \leq T}$ for Q_k it follows that $(Z_{s,t}^{a,b,M})_{s \leq t \leq T}$ is a martingale for Q .

4. step 4. Q_k is weakly converging to Q on Ω^s . X_s is continuous on Ω^s , thus for all $\eta > 0$, $\{||X_s - y|| > \eta\}$ is open. It follows that $Q(\{||X_s - y|| > \eta\}) \leq \liminf Q_k(\{||X_s - y|| > \eta\})$.

-If $s_n < s$, Q_k is the image of $Q_{s_{n_k}, y_{n_k}}^{a,b,M}$. let $\epsilon, \eta > 0$. Let h such that equation (11) is satisfied for all $Q_{s_{n_k}, y_{n_k}}^{a,b,M}$ and Q . Let k_0 such that for all $k \geq k_0$, $|s_{n_k} - s| < h$ and $||y - y_{n_k}|| < \eta$. It follows from (11) applied with $Q_{s_{n_k}, y_{n_k}}^{a,b,M}$, $u = s_{n_k}$ and $v = s$ that $Q_{s_{n_k}, y_{n_k}}^{a,b,M}(\{||X_s - y|| > 2\eta\}) < 2\epsilon$. Q is the weak limit of $Q_{s_{n_k}, y_{n_k}}^{a,b,M} (\pi_{[s,T]}^{s_{n_k}})^{-1}$, and s belongs to T_Q , it follows that $Q(\{X_s = y\}) = 1$.

- If s_n is decreasing to s , $Q_k = Q_{s_{n_k}, y_{n_k}}^{a,b,M}$. Let $\epsilon > 0$, for k large enough, $Q_k(\{||X_s - y|| > \epsilon\}) = 0$. In both cases this proves that $Q = Q_{s,y}^{a,b,M}$.

□

Proposition 2.10 *Let a and b be continuous bounded on $([0, T] \times \mathbb{R}^n)$ such that $a(s, x)$ is invertible for all (s, x) . Assume that M satisfies the hypothesis*

M_C . $Z_{r,t}^{a,b,M}$ given by equation (6) is bounded. The unique solution $Q_{r,y}^{a,b,M}$ on $(\mathcal{D}^r, \mathcal{B}_T^r)$ to the additive martingale problem $Z_{r,t}^{a,b,M}$ starting from y at time r is weakly continuous. The set

$$\Theta = \{(a, b, M) \mid a, b \text{ continuous bounded, } a \text{ invertible, } M \text{ satisfies hypothesis } M_C \text{ for some } C > 0\} \quad (15)$$

satisfies hypothesis H_Θ .

Proof The existence and unicity follow from [32]. The weak continuity follows then from Proposition 2.9 applied with $s_n = r$ for all n . As already noticed in step 3 of Proposition 2.9, for all $f \in \mathcal{C}_c^\infty$, and all $s \leq t$, $Z_{s,t}^{a,b,M}$ is bounded. \square .

This weak continuity for given r can also be found in [13] Lemma 5.2., but there is no detailed proof.

3 Canonical representation of the conditional expectation for the weakly continuous solution to a martingale problem

Let $0 \leq r < s \leq T$. The state space Ω_T^r is either $\mathcal{C}([r, T], \mathbb{R}^n)$ or $\mathcal{D}([r, T], \mathbb{R}^n)$. The goal of this Section is to prove that in the setting of weakly continuous solutions to martingale problems, there exists a canonical choice for a regular conditional probability, and that this choice satisfies nice properties.

3.1 Unicity of the solution to a martingale problem for non Markovian parameters

Let Θ be a set of parameters satisfying Hypothesis H_Θ (Definition 2.4). Given $r \in [0, T]$ and $y \in \mathbb{R}^n$, we introduce the following stable set of parameters:

Definition 3.1 Let $0 \leq r \leq T$. Θ_T^r denotes the set of \mathcal{B}_T^r predictable processes γ on $(\Omega^r, \mathcal{B}_T^r)$ such that:

There is a finite subdivision $r = s_0 < s_1 < \dots < s_n = T$.

For all $i \in \{0, 1, \dots, n-1\}$ there is a finite set I_i , $I_0 = \{1\}$, a finite partition $(A_{i,j})_{j \in I_i}$ of Ω^r into $\mathcal{B}_{s_i}^r$ -measurable sets, and $\theta_0, (\theta_{i,j})_{j \in I_i} \in \Theta$ such that

$$\forall s_i \leq u < s_{i+1}, \forall \omega \in \Omega^r, \gamma(u, \omega) = \sum_{j \in I_i} \theta_{i,j}(u, X_u(\omega)) 1_{A_{i,j}}(\omega) \quad (16)$$

We define the associated martingale:

Definition 3.2 For all γ in Θ_T^r admitting the above representation, denote $(Z_{r,u}^\gamma)_{r \leq u \leq T}$ the additive process $(Z_{r,v}^\gamma = Z_{r,u}^\gamma + Z_{u,v}^\gamma \text{ for } r \leq u \leq v \leq T)$ defined by

$$\forall s_i \leq u \leq s_{i+1}, \forall \omega \in \Omega, Z_{s_i,u}^\gamma(\omega) = \sum_{j \in I_i} Z_{s_i,u}^{\theta_{i,j}}(\omega) 1_{A_{i,j}}(\omega) \quad (17)$$

Lemma 3.3 Assume that $Q_{r,x}^\gamma$ is a solution to the bounded additive martingale problem $(Z_{r,t}^\gamma)_{r \leq t \leq T}$ starting from x at time r (Definition 2.3). Let $(Q_{r,x}^\gamma)^{s,\omega}$ be a regular conditional distribution of $Q_{r,x}^\gamma$ given \mathcal{B}_s^r . There is subset N of Ω_T^r such that for all $\omega \in \Omega_T^r - N$, $(Q_{r,x}^\gamma)^{s,\omega}$ is solution on Ω_T^r to the martingale problem $(Z_{s,t}^\gamma)_{s \leq t \leq T}$ and $(Q_{r,x}^\gamma)^{s,\omega}(\{X_u(\omega') = X_u(\omega), \forall r \leq u \leq s\}) = 1$.

Proof Let $s \leq u \leq v \leq t$. For all $B \in \mathcal{B}_u^r$, there is a set N_B such that for $Q_{r,x}^\gamma(N_B) = 0$ and for all $\omega \in \Omega_T^r - N_B$,

$$\begin{aligned} & (Q_{r,x}^\gamma)^{s,\omega}(Z_{s,v}^\gamma - Z_{s,u}^\gamma)1_B) \\ &= E_{Q_{r,x}^\gamma}((Z_{r,v}^\gamma - Z_{r,u}^\gamma)1_B | \mathcal{B}_s^r)) \\ & E_{Q_{r,x}^\gamma}(E_{Q_{r,x}^\gamma}((Z_{r,v}^\gamma - Z_{r,u}^\gamma)1_B | \mathcal{B}_u^r) | \mathcal{B}_s^r) = 0 \end{aligned} \quad (18)$$

The σ -algebra \mathcal{B}_u^r is a Borel σ -algebra. Thus it is countably generated. Let A_k be a countable family of events including Ω_T^r generating \mathcal{B}_u^r . It follows that there is a subset N of Ω_T^r such that $Q_{r,x}^\gamma(N) = 0$ and

$$(Q_{r,x}^\gamma)^{s,\omega}(Z_{s,v}^\gamma - Z_{s,u}^\gamma)\xi) = 0 \quad (19)$$

for all $\xi = 1_{A_k}$ and all $\omega \in \Omega_T^r - N$. It follows from the dominated convergence Theorem and the monotone class theorem that equation (19) is satisfied by 1_A for all A in \mathcal{B}_u^r . Thus equation (19) is satisfied for all ξ simple \mathcal{B}_u^r measurable. Another application of the dominated convergence Theorem proves that it is satisfied for all ξ bounded \mathcal{B}_u^r measurable. We have thus proved that for all $s \leq u \leq v \leq t$, there is a set $N_{u,v}$, $Q_{r,x}^\gamma(N_{u,v}) = 0$ such that equation (19) is satisfied for all ξ bounded \mathcal{B}_u^r measurable and all $\omega \in \Omega_T^r - N_{u,v}$. Let $N = \cup_{u,v \in (\mathcal{Q} \cap [s,T]) \cup \{s,T\}} N_{u,v}$. Making use of the rightcontinuity of the process $(Z_{r,u}^\gamma)_{r \leq u \leq t}$ for given r , equation (19) follows for all $s \leq u \leq v \leq t$, for all ξ bounded \mathcal{B}_u^r measurable and all $\omega \in \Omega_T^r - N$.

Thus for all $\omega \in \Omega_T^r - N(\cdot, Q_{r,x}^\gamma)^{s,\omega}$ is solution on Ω_T^r to the martingale problem $(Z_{s,t}^\gamma)_{s \leq t \leq T}$. By definition of a regular conditional distribution, $(Q_{r,x}^\gamma)^{s,\omega}(\{X_u(\omega') = X_u(\omega), \forall r \leq u \leq s\}) = 1$. \square

Proposition 3.4 *For all process γ in Θ_T^r , for all $0 \leq r < t \leq T$ there is at most one probability measure on $(\Omega_t^r, \mathcal{B}_t^r)$ solution to the additive martingale problem $(Z_{r,u}^\gamma)$ starting from y at time r .*

Proof Let γ in Θ_T^r (Definition 3.1). The proof is done by iteration on $n \geq 1$. For $n = 1$ it is satisfied by hypothesis on Θ . Assume that it is satisfied for $n - 1$. Let Q be a probability measure solution to the martingale problem $(Z_{r,u}^\gamma)$ on $(\Omega_t^r, \mathcal{B}_t^r)$ starting from y at time r . It is easy to see that the restriction of Q to $\mathcal{B}_{s_{n-1}}^r$ is solution to the martingale problem $(Z_{r,u}^{\tilde{\gamma}})_{r \leq u \leq s_{n-1}}$ where $\tilde{\gamma}$ is the restriction of γ to $[r, s_{n-1}] \times \Omega_t^r$. By recursion hypothesis, the restriction of Q to $\mathcal{B}_{s_{n-1}}^r$ is uniquely determined. Let $Q^{s_{n-1}, \omega}$ be a regular conditional distribution of Q given $\mathcal{B}_{s_{n-1}}^r$. From Lemma 3.3, for all $j \in I_{n-1}$, there is a subset N_j of $A_{n-1,j}$ such that for all $\omega \in A_{n-1,j} - N_j$, $Q^{s_{n-1}, \omega}$ is solution to the martingale problem $(Z_{s_{n-1},v}^{\theta_{n-1,j}})_{s_{n-1} \leq v \leq t}$. The function $\theta_{n-1,j}$ belongs to Θ , thus from unicity of the solution to the martingale problem associated to $\theta_{n-1,j}$ starting from $X_{s_{n-1}}(\omega)$ at time s_{n-1} , it follows that $Q_{s_{n-1}, \omega}$ is uniquely determined for all $\omega \in A_{n-1,j} - N_j$ with $Q(N_j) = 0$, for all $j \in I_{n-1}$. Thus Q is uniquely determined. Therefore there exists at most one probability measure Q solution to the martingale problem $(Z_{r,u}^\gamma)$ on $(\Omega_t^r, \mathcal{B}_t^r)$ starting from y at time r . \square

3.2 Construction of a probability measure \overline{Q} on $(\Omega^r, \mathcal{B}_T^r)$ associated to a non Markovian parameter

For every Borelian function f on $\Omega_s^r \times \Omega_T^s$, for all ω in Ω_s^r , f^ω denotes the Borelian map defined on Ω_T^s by $f^\omega(\omega') = f(\omega, \omega')$.

Lemma 3.5 *For given s , let $(Q_{s,y})_{y \in \mathbb{R}^n}$ be the weakly continuous solution on $(\Omega^s, \mathcal{B}_T^s)$ to a martingale problem starting from y at time s (Definition 2.3). Then for all f Borelian bounded on $\Omega_s^r \times \Omega_T^s$, the map*

$$\begin{aligned} \Omega_s^r \times \mathbb{R}^n &\rightarrow \mathbb{R} \\ (\omega, y) &\rightarrow Q_{s,y}(f^\omega) \end{aligned} \tag{20}$$

is measurable when $\Omega_s^r \times \mathbb{R}^n$ is endowed with the Borel product σ -algebra. When f is continuous on $\Omega_s^r \times \Omega_T^s$, the above map is furthermore separately continuous in each variable.

Proof

- Assume that f is continuous bounded. Being f^ω continuous, the map

$$y \rightarrow Q_{s,y}(f^\omega) \quad (21)$$

is continuous by hypothesis. On the other hand, $Q_{s,y}$ being a probability measure and f being continuous bounded, the continuity of the map

$$\omega \rightarrow Q_{s,y}(f^\omega). \quad (22)$$

follows from the dominated convergence Theorem. Thus the map

$$(\omega, y) \rightarrow Q_{s,y}(f^\omega) \quad (23)$$

is separately continuous in each variable. The sets Ω_s^r and Ω_T^s are metrizable and separable, it follows from Lemma 4.50 of [1] that this map is jointly measurable for the product Borel σ -algebra.

- Let \mathcal{H} be the set of bounded Borelian functions f on $\Omega_s^r \times \Omega_T^s$ such that the map $(\omega, y) \rightarrow Q_{s,y}(f^\omega)$ is Borelian. \mathcal{H} is a vector space containing the constant functions. If f_n is an increasing sequence of non negative functions in \mathcal{C} with limit f bounded, it follows from the monotone convergence theorem that $Q_{s,y}(f^\omega) = \lim_{n \rightarrow \infty} Q_{s,y}(f_n^\omega)$. It follows that f belongs to \mathcal{H} . From the first step of the proof \mathcal{H} contains the class \mathcal{C} of continuous bounded functions. The class \mathcal{C} is stable by pointwise multiplication. It follows from the monotone class theorem as stated in [29] chapter 0 Theorem 2.2, that \mathcal{H} contains all the bounded Borelian functions on $\Omega_s^r \times \Omega_T^s$. \square

Corollary 3.6 *For all f Borelian bounded on $\Omega_s^r \times \Omega_T^s$, the map*

$$\Omega_s^r \rightarrow \mathbb{R} \quad (24)$$

$$\omega \rightarrow Q_{s,X_s(\omega)}(f^\omega) \quad (25)$$

is Borelian.

Proof The map $\omega \in \Omega_s^r \rightarrow X_s(\omega) \in \mathbb{R}^n$ is Borelian, thus the corollary follows from Lemma 3.5. \square

Proposition 3.7 *Let $0 \leq r < s$. Let J be a finite subset of \mathbb{N} . For given $j \in J$, let $(Q_{s,y}^j)_{y \in \mathbb{R}^n}$ be a weakly continuous family of probability measures on Ω_T^s . Let $(h_j)_{j \in J}$ be non negative Borelian maps on \mathbb{R}^n such that $\forall y \in \mathbb{R}^n, \sum_{j \in J} h_j = 1$. Let Q_1 be a probability measure on $(\Omega_s^r, \mathcal{B}_s^r)$. There is a unique probability measure \tilde{Q} on $(\Omega_s^r \times \Omega_T^s, \mathcal{B}_s^r \times \mathcal{B}_T^s)$ such that for all f Borelian bounded on $\Omega_s^r \times \Omega_T^s$,*

$$\tilde{Q}(f) = \sum_{j \in J} \int_{\Omega_s^r} h_j(X_s(\omega)) Q_{s,X_s(\omega)}^j(f^\omega) dQ_1(\omega) \quad (26)$$

Proof For all f Borelian bounded it follows from Corollary 3.6 that the map

$$T(f) : \omega \in \Omega_s^r \rightarrow \sum_{j \in J} h_j(X_s(\omega)) Q_{s,X_s(\omega)}^j(f^\omega) \quad (27)$$

is Borelian. It is also bounded. Thus $\int_{\Omega_s^r} \sum_{j \in J} h_j(X_s(\omega)) Q_{s,X_s(\omega)}^j(f^\omega) dQ_1(\omega)$ is well defined. The map

$$L : f \rightarrow \sum_{j \in J} \int_{\Omega_s^r} h_j(X_s(\omega)) Q_{s,X_s(\omega)}^j(f^\omega) dQ_1(\omega) \quad (28)$$

defines a non negative linear form on the vector space of continuous bounded functions on $\Omega_s^r \times \Omega_T^s$. Let f_n be a decreasing sequence of continuous bounded functions on $\Omega_s^r \times \Omega_T^s$ converging to 0. From the dominated convergence Theorem, it follows that for all j and ω , $Q_{s,X_s(\omega)}^j(f_n^\omega)$ is a decreasing bounded sequence with limit 0. Applying again the dominated convergence Theorem, it follows that $L(f_n)$ is decreasing with limit 0. From Daniell Stone Theorem it follows that there is a unique probability measure \tilde{Q} on $(\Omega_s^r \times \Omega_T^s, \mathcal{B}_s^r \times \mathcal{B}_T^s)$, such that for all f continuous bounded, equation (26) is satisfied. The equality (26) follows then for all Borelian bounded function as in the end of the proof of Lemma 3.5. \square

Definition 3.8 *Denote \bar{Q} the probability measure on Ω_T^r deduced from \tilde{Q} by the Borelian map*

$$\begin{aligned} i : \Omega_s^r \times \Omega_T^s &\rightarrow \Omega_T^r \\ (\omega, \omega') &\rightarrow i(\omega, \omega') \end{aligned} \quad (29)$$

where

$$\begin{aligned} i(\omega, \omega')(u) &= \omega(u), \text{ for } r \leq u \leq s \\ &= \omega'(u) + \omega(s) - \omega'(s), \text{ for } s \leq u \leq T \end{aligned} \quad (30)$$

3.3 Canonical regular conditional probability for \overline{Q} given \mathcal{B}_s^r

Recall that Π_s^r denotes the canonical projection from Ω_T^r to Ω_s^r . (cf Notation 12).

Proposition 3.9 • For all ψ Borelian bounded on Ω_T^r , for all ω in Ω_T^r , let

$$T_s(\psi)(\omega) = \sum_{j \in J} h_j(X_s(\omega))(Q^j)_{s, X_s(\omega)}((\psi \circ i)^{\pi_s^r(\omega)}) \quad (31)$$

The map $T_s(\psi)$ defined on Ω_T^r is \mathcal{B}_s^r measurable.

- For all \tilde{Q} , and all ψ Borelian bounded, $T_s(\psi)$ is a \mathcal{B}_s^r measurable version of the \overline{Q} conditional expectation of ψ given \mathcal{B}_s^r .
For all g \mathcal{B}_s^r measurable, $T_s(g) = g$.

Proof

- The map i is measurable for the Borel σ -algebras on both side, it follows from Lemma 3.5 that for every Borelian map ψ on Ω_T^r , the map

$$\begin{aligned} \Omega_s^r \times \mathbb{R}^n &\rightarrow \mathbb{R} \\ (\omega, y) &\rightarrow Q_{s,y}^j((\psi \circ i)^\omega) \end{aligned} \quad (32)$$

is Borelian for all j . Notice that the map π_s^r is measurable from Ω_T^r endowed with the σ -algebra \mathcal{B}_s^r into Ω_s^r endowed with its Borel σ -algebra. Composing the above map (32) with

$$\begin{aligned} \Omega_T^r \times \mathbb{R}^n &\rightarrow \Omega_s^r \times \mathbb{R}^n \\ (\omega, y) &\rightarrow (\pi_s^r(\omega), y) \end{aligned} \quad (33)$$

it follows that

$$\begin{aligned} \Omega_T^r \times \mathbb{R}^n &\rightarrow \mathbb{R} \\ (\omega, y) &\rightarrow Q_{s,y}^j((\psi \circ i)^{\pi_s^r(\omega)}) \end{aligned} \quad (34)$$

is measurable for the σ -algebra product of \mathcal{B}_s^r and of $\mathcal{B}(\mathbb{R}^n)$. h_j a Borelian map on \mathbb{R}^n and X_s is \mathcal{B}_s^r measurable, it follows that $T_s(\psi)$ is \mathcal{B}_s^r measurable for all Borelian map ψ .

- Notice that every \mathcal{B}_s^r measurable function g defined on Ω_T^r can be factorized $g = \tilde{g} \circ \pi_s^r$ for some \tilde{g} \mathcal{B}_s^r measurable defined on Ω_s^r . This result is deduced from the monotone class theorem ([29]) applied with the class \mathcal{C} of functions $f(X_{t_1}, X_{t_2}, \dots, X_{t_k})$, $k \in \mathbb{N}^*$, f Borelian on $(\mathbb{R}^n)^k$, $r \leq t_1 \leq t_2 \leq t_k \leq s$. For all ω in Ω_T^r and ω' in Ω_T^s , $\pi_s^r \circ i(\pi_s^r(\omega), \omega') = \pi_s^r(\omega)$. It follows that for all g \mathcal{B}_s^r measurable, the map $(g \circ i)^{\pi_s^r(\omega)}$ is constant equal to $g(\omega)$. From the equality $\sum_{j \in J} h_j = 1$, it follows that for all g \mathcal{B}_s^r measurable, $T_s(g) = g$. $T_s(\psi)$ factorizes, $T_s(\psi) = \tilde{T}_s(\psi) \circ \pi_s^r$ where $\tilde{T}_s(\psi)$ is defined on Ω_s^r by

$$\tilde{T}_s(\psi)(\omega) = \sum_{j \in J} h_j(X_s(\omega)) Q_{s, X_s(\omega)}^j(\psi \circ i)^\omega \quad (35)$$

From the definition of \bar{Q} (Proposition 3.7 and Definition 3.8), it follows that for all Ψ \mathcal{B}_T^r measurable, and $g = \tilde{g} \circ \pi_s^r$ \mathcal{B}_s^r measurable defined on Ω_T^r ,

$$\bar{Q}(\psi g) = \sum_{j \in J} \int_{\Omega_s^r} (\tilde{g})(\omega) h_j(X_s(\omega)) Q_{s, X_s(\omega)}^j(\psi \circ i)^\omega dQ_1(\omega) \quad (36)$$

From the equality $T_s(\psi) = \tilde{T}_s(\psi) \circ \Pi_s^r$, it follows that

$$\bar{Q}(T_s(\psi)g) = \int_{\Omega_s^r} (\tilde{g})(\omega) (\tilde{T}_s(\psi)(\omega)) dQ_1(\omega) \quad (37)$$

The map $T_s(\psi)$ being \mathcal{B}_s^r measurable, it follows from equations (36), (35) and (37) that $T_s(\psi)$ is a \mathcal{B}_s^r measurable version of the \bar{Q} conditional expectation of ψ given \mathcal{B}_s^r . \square

3.4 Existence of a solution to the martingale problem for non Markovian parameters

The goal of this Section is to prove that for all $\gamma \in \Theta_T^r$ (Definition 3.1), the martingale problem Z^γ introduced in Definition 3.2 has a solution.

Proposition 3.10 *The hypothesis are the same as in Proposition 3.7. Assume furthermore that $Q^1 = Q_{r,x}^1$ is the unique solution on Ω_s^r to the $(Z_t^\theta)_{r \leq t}$ martingale problem starting from x at time r . Assume that for all $j \in \bar{J}$, $Q_{s,y}^j$ is the unique solution on Ω_T^s to the $(Z_t^{\theta_j})_{s \leq t}$ martingale problem starting from y at time s . Assume that there is a finite partition $(A_j)_{j \in J}$ of Ω^r in \mathcal{B}_s^r measurable sets such that for all $j \in J$, $h_j = 1_{A_j}$. Assume that for all*

j in J , $(Q_{s,y}^j)_{y \in \mathbb{R}^n}$ is weakly continuous. The probability measure \overline{Q} is the unique solution to the Z^β martingale problem starting from x at time r with $\beta(u, x) = \theta(u, x)$, for all $r \leq u < s$ and $\beta(u, x) = \sum_{j \in J} 1_{A_j} \theta_j(u, x)$, for all $s \leq u$.

Proof

- Let $s \leq t < u \leq T$. By definition of β and the additive property of the martingales, $Z_{r,u}^\beta - Z_{r,t}^\beta = Z_{t,u}^\beta = \sum_{j \in J} 1_{A_j} (Z_{s,u}^{\theta_j} - Z_{s,t}^{\theta_j})$. For all ω in Ω_s^r and ω' in Ω_T^s , such that $X_s(\omega) = X_s(\omega')$, for all $s \leq v \leq T$, $i(\omega, \omega')(v) = \omega'(v)$, and $[(Z_{s,u}^{\theta_j} - Z_{s,t}^{\theta_j}) \circ i]^\omega(\omega') = (Z_{s,u}^{\theta_j} - Z_{s,t}^{\theta_j})(\omega')$. Notice also that for all ψ \mathcal{B}_t^r measurable and all ω in Ω_s^r , the map $(\psi \circ i)^\omega$ is \mathcal{B}_t^s measurable. Thus for all ψ \mathcal{B}_t^r measurable

$$\begin{aligned} & 1_{A_j}(X_s(\omega)) Q_{s, X_s(\omega)}^j [(\psi(Z_{r,u}^\beta - Z_{r,t}^\beta) \circ i)^\omega] \\ &= 1_{A_j}(X_s(\omega)) Q_{s, X_s(\omega)}^j [1_{\{\omega' \mid X_s(\omega) = X_s(\omega')\}} (\psi \circ i)^\omega [(Z_{s,u}^{\theta_j} - Z_{s,t}^{\theta_j}) \circ i]^\omega] \\ &= 1_{A_j}(X_s(\omega)) Q_{s, X_s(\omega)}^j [(\psi \circ i)^\omega (Z_{s,u}^{\theta_j} - Z_{s,t}^{\theta_j})] \quad (38) \end{aligned}$$

$(Z_{s,t}^{\theta_j})_{s \leq t \leq T}$ is a $Q_{s,y}^j$ martingale for all y , it follows that the last term of (38) is equal to 0. It follows then from (38) and the definition of \overline{Q} (Section 3.2) that $Z_{r,t}^\beta$ is a \overline{Q} martingale for $s \leq t \leq T$.

- Let $r \leq t < u \leq s$. $Z_{r,u}^\beta - Z_{r,t}^\beta = (Z_{r,u}^\theta - Z_{r,t}^\theta) = (Z_{r,u}^\theta - Z_{r,t}^\theta) \circ (\pi_s^r)$. Let ψ be a function \mathcal{B}_t^r measurable on Ω_T^r , $\psi = \tilde{\psi} \circ (\pi_s^r)$, where $\tilde{\psi}$ is a function \mathcal{B}_t^r measurable defined on Ω_s^r . Being ψ and $Z_{r,u}^\beta - Z_{r,t}^\beta$ \mathcal{B}_s^r measurable, it follows from the second part of the proof of Proposition 3.9 that $\overline{Q}(\psi(Z_{r,u}^\beta - Z_{r,t}^\beta)) = Q_1(\tilde{\psi}(Z_{r,u}^\theta - Z_{r,t}^\theta))$. Being $(Z_{r,t}^\theta)_{r \leq t \leq s}$ a Q_1 martingale, it follows that $Z_{r,t}^\beta$ is a \overline{Q} martingale for $r \leq t \leq s$. \square

3.5 Canonical regular conditional probability

Notation 3.11 For all $r \leq s$, for all $\omega \in \Omega_T^r$ and $\omega' \in \Omega_t^s$ such that $X_s(\omega) = X_s(\omega')$ we denote $\omega * \omega'$ the element of Ω_T^r such that

$$\begin{aligned} \omega * \omega'(u) &= \omega(u) \quad \forall r \leq u \leq s \\ \omega * \omega'(u) &= \omega'(u) \quad \forall s < u \leq T \end{aligned} \quad (39)$$

Proposition 3.12 1. Let $r \geq 0$ and y in \mathbb{R}^n . Let $\theta \in \Theta$. For all $s \in [r, T[$ and all bounded map ψ \mathcal{B}_T^r -measurable defined on Ω^r , the map $T_s^\theta(\psi)$ defined on Ω^r by $T_s^\theta(\psi)(\omega) = Q_{s, X_s(\omega)}^\theta((\psi \circ i)^{\pi_s^r(\omega)})$ is a \mathcal{B}_s^r -measurable version of the $Q_{r,y}^\theta$ -conditional expectation of ψ given \mathcal{B}_s^r . We call it the canonical regular conditional distribution of $Q_{r,y}^\theta$.

2.

$$T_s^\theta(\psi)(\omega) = Q_{s, X_s(\omega)}^\theta((\psi \circ i)^{\pi_s^r(\omega)}) \quad (40)$$

$$= \int_{\Omega_T^s} \psi(\omega * \omega') 1_{X_s(\omega) = X_s(\omega')} dQ_{s, X_s(\omega)}^\theta(\omega') \quad (41)$$

3. It satisfies the following chain rule for all map ϕ $\mathcal{B}_T^{r_0}$ -measurable defined on Ω^{r_0} ,

$$\forall r_0 \leq r \leq s \leq T, \quad T_r^\theta(\phi) = T_r^\theta(T_s^\theta(\phi)) \quad (42)$$

Proof

1. Apply Proposition 3.10 with $Q_1 = Q_{r,y}^\theta$, $J = \{j\}$, and $Q_{s,x}^j = Q_{s,x}^\theta$. It follows that the probability measure \overline{Q} satisfies the Z^θ martingale problem on Ω_T^r starting from y at time r . By unicity of the solution to the martingale problem it follows that $\overline{Q} = Q_{r,y}^\theta$. The result follows then from Proposition 3.9.

2. It follows immediately from the definitions of i , Π_s^r and $*$ that

$$1_{\{X_s(\omega) = X_s(\omega')\}} (\psi \circ i)^{\pi_s^r(\omega)}(\omega') = 1_{\{X_s(\omega) = X_s(\omega')\}} \psi(\omega * \omega') \quad (43)$$

The equation (41) follows then from $Q_{s, X_s(\omega)}^\theta(1_{\{X_s(\omega) = X_s(\omega')\}}) = 1$

3. Let \overline{Q} be as in 1.. From the definition of \overline{Q} , Definition 3.8, equations (26) and (41) it follows from the equality $\overline{Q} = Q_{r,y}^\theta$ that for all function ψ \mathcal{B}_T^r -measurable defined on Ω^r ,

$$\int_{\Omega_T^r} \psi(\omega') dQ_{r,y}^\theta(\omega') = \int_{\Omega_s^r} \left[\int_{\Omega_T^s} \psi(\omega_1 * \omega_2) 1_{X_s(\omega_1) = X_s(\omega_2)} dQ_{s, X_s(\omega_1)}^\theta(\omega_2) \right] dQ_{r,y}^\theta(\omega_1) \quad (44)$$

Let $\phi \mathcal{B}_T^{r0}$ measurable defined on Ω^{r0} . Let $\omega \in \Omega^{r0}$. Applying the above equation to the function ψ defined on Ω^r by $\psi(\omega') = \phi(\omega * \omega')1_{\{X_r(\omega)=X_r(\omega')\}}$, it follows from equation (41) that

$$\begin{aligned} T_r^\theta(\phi) &= \int_{\Omega_T^r} \phi(\omega * \omega')1_{X_r(\omega)=X_r(\omega')}dQ_{r,X_r(\omega)}^\theta(\omega') = \\ &= \int_{\Omega_T^r} 1_{X_r(\omega)=X_r(\omega_1*\omega_2)}dQ_{r,X_r(\omega)}^\theta(\omega_1)\left[\int_{\Omega_T^s} \phi(\omega * \omega_1 * \omega_2)1_{X_s(\omega_1)=X_s(\omega_2)}dQ_{s,X_s(\omega_1)}^\theta(\omega_2)\right] \\ &= \int_{\Omega_T^r} 1_{X_r(\omega)=X_r(\omega_1)T_s^\theta(\phi)(\omega*\omega_1)}dQ_{r,X_r(\omega)}^\theta(\omega_1) \end{aligned} \quad (45)$$

This proves the chain rule equation (42). \square

Theorem 3.13 *Let $0 \leq r \leq T$. For all $\gamma \in \Theta_T^r$ and y in \mathbb{R}^n , there is a unique solution to the martingale problem Z^γ on $(\Omega^r, \mathcal{B}_T^r)$ starting from y at time r . We denote it $Q_{r,y}^\gamma$. Let $r \leq s < T$. Consider the expression of γ as in Definition 3.1. For all $r \leq s < T$, let i such that $s_i \leq s < s_{i+1}$. For all ψ defined on Ω^r bounded $\mathcal{B}_{s_{i+1}}^r$ measurable, the map $T_s^\gamma(\psi)$ defined as*

$$T_s^\gamma(\psi) = \sum_{j \in J_i} 1_{A_{i,j}} T_s^{\theta_{i,j}}(\psi) \quad (46)$$

is $\mathcal{B}_{s_i}^r$ measurable. The map T_s^γ admits a unique extension to all bounded \mathcal{B}_T^r measurable functions such that the chain rule is satisfied:

$$\forall r \leq s \leq t \leq T, T_s^\gamma(\psi) = T_s^\gamma(T_t^\gamma(\psi)) \quad (47)$$

$T_s^\gamma(\psi)$ is a \mathcal{B}_s^r measurable version of the $Q_{r,y}^\gamma$ conditional expectation of ψ given \mathcal{B}_s^r . It is called the canonical regular conditional distribution of $Q_{r,y}^\gamma$ given \mathcal{B}_s^r .

Proof For given r and y the existence of $Q_{r,y}^\gamma$ follows by induction from Proposition 3.10. The unicity follows from Proposition 3.4. Let s_i be as subdivision asociated to γ as in Definition 3.1. Let $s_i \leq s < s_{i+1} \leq T$. Consider the probability measures \overline{Q} constructed on $\Omega_{s_{i+1}}^r$ from $Q_{r,y}^\gamma$ on Ω_s^r and from $\sum 1_{A_{i,j}}(\omega)Q_{s,X_s(\omega)}^{\theta_{i,j}}$ on Ω_T^s as in Section 3.2. From Proposition 3.10 it follows that \overline{Q} solves the martingale problem for γ on $\Omega_{s_{i+1}}^r$ starting from y at time r . From the unicity it follows that the restriction of $Q_{r,y}^\gamma$ to $\mathcal{B}_{s_{i+1}}^r$ is equal to \overline{Q} . From Proposition 3.9 equation (31) and equation (40), it follows that for all $\psi \mathcal{B}_{s_{i+1}}^r$ measurable, $T_s^\gamma(\psi)$ defined by equation (46) is a

\mathcal{B}_s^r measurable version of the \overline{Q} - conditional expectation of ψ given \mathcal{B}_s^r . It is thus also a \mathcal{B}_s^r measurable version of the $Q_{r,y}^\gamma$ -conditional expectation of ψ given \mathcal{B}_s^r .

Making use of the chain rule for T^θ , Proposition 3.12, it follows that T_s^γ admits a unique extension to all \mathcal{B}_T^r measurable functions such that the chain rule (47) is satisfied. It is given by a stepwise evaluation: for $s_i \leq s < s_{i+1}$, $T_s^\gamma(\psi) = T_s^\gamma(T_{s_{i+1}}^\gamma(\dots T_{s_{n-1}}^\gamma(\psi)))$. The result follows then by induction. \square

Corollary 3.14 *1. Let $r \geq 0$, for all $\gamma \in \Theta_T^r$, and $r \leq s < T$, the non negative linear map T_s^γ is continuous from below. It can be uniquely extended to the Borelian functions bounded from below by the formula*

$$T_s^\gamma(\psi) = \lim_{n \rightarrow \infty} T_s^\gamma(\psi \wedge n) \quad (48)$$

This extension is continuous from below.

- 2. Equation (41) (resp (46)) is still satisfied for random variables bounded from below, for all $\theta \in \Theta$ (resp $\gamma \in \Theta_T^r$).*
- 3. $T_s^\gamma(\psi)$ is a \mathcal{B}_s^r -measurable version of the $Q_{r,y}^\gamma$ -conditional expectation of ψ given \mathcal{B}_s^r .*
- 4. The chain rule is satisfied for all function ψ \mathcal{B}_T^r -measurable bounded from below,*

$$\forall r \leq s \leq t \leq T, \quad T_s^\gamma(\psi) = T_s^\gamma(T_t^\gamma(\psi)) \quad (49)$$

Proof The continuity from below for T_s^θ , $\theta \in \Theta$ follows from equation (41) and from the monotone convergence theorem applied to the probability measure $Q_{s,X_s(\omega)}^\theta$ for given ω . It follows for general T_s^γ from the formula (46). The other statements follow from the continuity from below of T_s^γ and from the monotone convergence theorem for probability measures (or for conditional expectations to prove 3.). \square

We are now able to give a definition of a stable set of probability measures for probability measures which are non dominated by some probability measure.

4 Stable set of probability measures and penalties

We concentrate now on the non dominated framework. As already noticed the usual definition of a stable set of probability measures cannot be used because the Q -conditional expectation is defined up to Q -null set. The existence of a canonical choice for the regular conditional probability is crucial in the construction of dynamic procedures in non dominated setting.

4.1 Stable set of probability measures

Let $0 \leq r \leq T$. Let Ω^r be \mathcal{C}^r or \mathcal{D}^r with the canonical filtration. For all $r \leq s \leq T$, denote $(\mathcal{B}_b)_s^r$ the set of bounded functions defined on Ω^r , \mathcal{B}_s^r -measurable. Let \mathcal{T} be a subset of $[r, T]$.

Definition 4.1 A set \mathcal{Q} of probability measures on $(\Omega^r, \mathcal{B}_T^r)$ is \mathcal{T} stable if

1. *Special choice of a regular conditional distribution*

For all $Q \in \mathcal{Q}$ for all $r \leq s \leq T$ $s \in \mathcal{T}$, there is a non negative linear map continuous from below $T_s^Q : (\mathcal{B}_b)_T^r \rightarrow (\mathcal{B}_b)_s^r$ such that for all $\psi \in (\mathcal{B}_b)_T^r$, $T_s^Q(\psi)$ is a version of the Q -conditional expectation of ψ given $(\mathcal{B}_b)_s^r$.

2. *Chain rule*

For all map ψ in $(\mathcal{B}_b)_T^r$, for all s, t in \mathcal{T} , $s \leq t$

$$T_s^Q(\psi) = T_s^Q(T_t^Q(\psi)) \quad (50)$$

3. *Stability by composition*

For all Q and R in \mathcal{Q} , for all $s \in \mathcal{T}$ there is a probability measure S in \mathcal{Q} such that

$$\begin{aligned} \forall s \leq u \leq T, T_u^S &= T_u^R \\ \forall r \leq u < s, \forall \phi \in (\mathcal{B}_b)_s^r, T_u^S(\phi) &= T_u^Q(\phi) \end{aligned} \quad (51)$$

4. *stability by bifurcation:*

for all $s \in \mathcal{T}$ for all Q and R in \mathcal{Q} , for all $A \subset \Omega^r$, $A \in \mathcal{B}_s^r$, there is a probability measure S in \mathcal{Q} such that

$$\forall s \leq u \leq T, T_u^S = 1_A T_u^Q + 1_{A^c} T_u^R \quad (52)$$

In all the following to simplify the notations we assume that $\mathcal{T} = [r, T]$.

Theorem 4.2 Assume that Θ satisfies hypothesis H_Θ (Definition 2.4).

Let $0 \leq r \leq T$ and y in \mathbb{R}^n . Let Θ_T^r be the set introduced in Definition 3.1. The set

$$\mathcal{Q}_{r,y}^\Theta = \{Q_{r,y}^\gamma, \gamma \in \Theta_T^r\} \quad (53)$$

is a stable set of probability measures.

More precisely, let $Q = Q_{r,y}^\gamma$ and $R = Q_{r,y}^\delta$ in $\mathcal{Q}_{r,y}^\Theta$.

i) Let $\lambda(u, \omega) = \gamma(u, \omega)$ for all $r \leq u < s$ and $\lambda(u, \omega) = \delta(u, \omega)$ for all $s \leq u < T$. The probability measure $S = Q_{r,y}^\lambda$ satisfies (51).

ii) Let $\eta(u, \omega) = \gamma(u, \omega)$ for all $r \leq u < s$ and $\eta(u, \omega) = 1_A(\omega)\gamma(u, \omega) + 1_{A^c}\delta(u, \omega)$ for all $s \leq u < T$. The probability measure $S = Q_{r,y}^\eta$ satisfies (52).

Proof For $\gamma \in \Theta_T^r$ and $r \leq s < T$, let $T_s^{Q_{r,y}^\gamma} = T_s^\gamma$. It follows from Corollary 3.14 that $T_s^{Q_{r,y}^\gamma}$ is linear non negative and continuous from below. Properties 1. and 2. follow from Theorem 3.13.

- Proof of property 3. The process λ defined in i) belongs to Θ_T^r and from Theorem 3.13 there is a unique probability measure $Q_{r,y}^\lambda$ solution on $(\Omega^r, \mathcal{B}_T^r)$ to the martingale problem associated to λ starting from y at time r . It follows also from equation (46) and the chain rule that for all $r \leq u \leq v \leq T$ for all ψ defined on Ω^r \mathcal{B}_v^r measurable, the \mathcal{B}_u^r measurable map $T_u^\lambda(\psi)$ depends only on the restriction of λ to $[u, v]$. This end the proof of the stability by composition.

- Property 4. is proved in the same way considering the process η defined in ii) \square

As in [5] we can furthermore consider a multivalued Borel mapping Γ from $\mathbb{R}^+ \times \mathbb{R}^n$ into E (definition 2.8 of [5]). For all (t, x) , $\Gamma(t, x)$ is a subset of E containing 0. Let $\Theta(\Gamma) = \{\theta \in \Theta \mid \theta(t, x) \in \Gamma(t, x) \forall (t, x)\}$. We then get the following corollary

Corollary 4.3 *Assume that Θ satisfies hypothesis H_Θ . For all multivalued Borel mapping Γ , the set*

$$\mathcal{Q}^\Theta(\Gamma)_{r,y} = \{Q_{r,y}^\gamma, \gamma \in \Theta(\Gamma)_T^r\} \quad (54)$$

is a stable set of probability measures. We call it the stable set of probability measures generated by $\Theta(\Gamma)$.

4.2 penalties

We want to construct time consistent dynamic convex Feller procedures, (and not only sublinear). Therefore as for the construction of time consistent dynamic risk measures on a filtered probability space, we need to introduce penalties satisfying some properties ([3]). To encompass the non dominated framework, we must adapt the definition of the required properties for penalties.

Definition 4.4 Assume that Θ satisfies hypothesis H_Θ . Let Γ be a multivalued Borel mapping. Let $\mathcal{Q} = \mathcal{Q}^\Theta(\Gamma)_{r,y}$. A penalty defined on \mathcal{Q} is a family $(\alpha_{s,t}(Q))_{r \leq s \leq t \leq T}$ of \mathcal{B}_s^r -measurable functions on Ω^r \mathcal{B}_s^r -measurable bounded from above such that

1. It is local:

For all $\gamma, \eta \in \Theta(\Gamma)_T^r$, for all $A \in \mathcal{B}_s^r$, if $1_A \gamma(u, \omega) = 1_A \eta(u, \omega)$, for all $u \in [s, t[$, then $1_A \alpha_{s,t}(Q_{r,y}^\gamma) = 1_A \alpha_{s,t}(Q_{r,y}^\eta)$.

2. It satisfies the cocycle condition:

$$\forall Q \in \mathcal{Q}, \forall r \leq s \leq t \leq u \leq T, \alpha_{s,u}(Q) = \alpha_{s,t}(Q) - T_s^Q(-\alpha_{t,u}(Q)) \quad (55)$$

Remark 4.5 Penalties could be defined for more general stable sets of probability measures.

The construction that we have made in [5] in the setting of (equivalent) probability measures solution to a martingale problem associated to a continuous diffusion leads also to penalties in the non dominated framework. This will be detailed when we construct time consistent dynamic procedures (Section 7).

Recall also that one of our goals is to construct time consistent convex procedures which give rise to viscosity solutions to second order partial differential equations. Therefore as in [5] we are interested in continuity properties.

5 Continuity properties for the canonical conditional probability

5.1 A continuity property in the case of continuous paths

Proposition 5.1 Case of \mathcal{C} : For all u in \mathbb{R}^+ and $x \in \mathbb{R}^n$, let $Q_{u,x}$ be the unique solution to a martingale problem. Assume that for some s , $(Q_{s,y})_{y \in \mathbb{R}^n}$ is weakly continuous.

1. Then for all $0 \leq r < s$ for all f bounded uniformly continuous (for the uniform norm) on Ω_T^r , the canonical conditional probability $T_s^{Q_{r,x}}(f)$:

$$\begin{aligned} \Omega_T^r &\rightarrow \mathbb{R} \\ (\omega &\rightarrow Q_{s,X_s(\omega)}((f \circ i)^{\pi_s^r(\omega)}) = Q_{s,X_s(\omega)}((f(\omega * \omega')1_{X_s(\omega)=X_s(\omega')}) \end{aligned} \quad (56)$$

is a continuous version of the $Q_{r,x}$ conditional expectation given \mathcal{B}_s^r .

2. If $f = g \circ \pi_{t'}^r$ for some g continuous on $\Omega_{t'}^r$ with compact support. Let $t'' = \inf(s, t')$. Then $Q_{s, X_s(\omega)}((f \circ i)^{\pi_s^r(\omega)}) = h \circ \pi_{t''}^r$ for some continuous function h on $\Omega_{t''}^r$ with compact support.

Proof

1. The map i introduced in Definition 3.8 is uniformly continuous for the uniform norm, it follows that for all f uniformly continuous bounded on Ω_T^r , the map $f \circ i$ is uniformly continuous bounded on the product space $\Omega_s^r \times \Omega_T^r$ for the uniform norm. Being $f \circ i$ uniformly continuous, for all $\epsilon > 0$ there is $\eta > 0$ such that for $\|\omega - \omega'\| < \eta$, $\|(f \circ i)^\omega - (f \circ i)^{\omega'}\| < \epsilon$. The continuity of the map

$$\begin{aligned} \Omega_s^r \times \mathbb{R}^n &\rightarrow \mathbb{R} \\ (\omega, y) &\rightarrow Q_{s,y}((f \circ i)^\omega) \end{aligned} \quad (57)$$

follows then easily from the weak continuity of the family $(Q_{s,y})_{y \in \mathbb{R}^n}$. being X_s continuous from Ω_T^r to \mathbb{R}^n and π_s^r continuous from Ω_T^r to Ω_s^r , the continuity follows by composition. From Proposition 3.12, equation (56) describes the canonical regular conditional distribution.

2. Assume now that $f = g \circ \pi_{t'}^r$ for some g continuous on $\Omega_{t'}^r$ with compact support K .
- (a) If $t' \leq s$, f is \mathcal{B}_s^r measurable, and from Proposition 3.9, $T_s^{Q_{r,x}}(f) = f$
- (b) If $s \leq t'$, then $t'' = s$. The maps g and $\pi_{t'}^r$ are uniformly continuous thus from 1., $T_s^{Q_{r,x}}(f)$ is continuous and \mathcal{B}_s^r measurable. Therefore it can be written $T_s^{Q_{r,x}}(f) = h \circ \pi_s^r$ for some continuous function h . The map $\pi : \omega \in \Omega_{t'}^r \rightarrow \omega|_{[r,s]} \in \Omega_s^r$ is continuous. Thus $\pi(K)$ is a compact subspace of Ω_s^r . If $\omega \notin \pi(K)$ and $\omega' \in \Omega_T^r$, $Q_{s, X_s(\omega)}((f(\omega * \omega')1_{X_s(\omega)=X_s(\omega')}) = 0$. This proves the result with the support of h contained in $\pi(K)$.

□

Notice that this provides a class of continuous functions stable by T^θ :

Notation 5.2 In case $\Omega^r = \mathcal{C}^r$, we denote

$$\mathcal{V}_t^r = \{f \circ \pi_u^r, r \leq u \leq t, f \text{ continuous on } \Omega_u^r \text{ with compact support} \} \quad (58)$$

For all $Y \in \mathcal{V}_t^r$, for all $y \in \mathbb{R}^k$ and $s \geq r$, $T_s^{Q_{r,y}}(Y)$ belongs to \mathcal{V}_t^r

Lemma 5.3 *Let Ω be a Polish space. Let \mathcal{Q} be a set of probability measures on $(\Omega, \mathcal{B}(\Omega))$. Assume that \mathcal{Q} is tight. For all f continuous bounded on C^r , there is an increasing sequence of continuous functions with compact support f_n such that $f = \lim_{n \rightarrow \infty} f_n$ Q a.s. $\forall Q \in \mathcal{Q}$.*

Proof By hypothesis there is an increasing sequence \mathcal{K}_n of compact subspaces of Ω such that $Q(\mathcal{K}_n^c) < \frac{1}{n}$ for all $Q \in \mathcal{Q}$. Let ϕ_n be an increasing sequence of continuous functions with compact support, $1_{\mathcal{K}_n} \leq \phi_n \leq 1$. Assume that f is continuous bounded, let $g = \lim_{n \rightarrow \infty} f\phi_n$. $0 \leq \|f - f\phi_n\| \leq \|f\|$. $Q(|f - f\phi_n|) \leq \|f\| \frac{1}{n}$. From the dominated convergence Theorem, it follows that $f = g = \lim f\phi_n$ Q a.s. $\forall Q \in \mathcal{Q}$.

5.2 Feller property

Definition 5.4 1. Let s in \mathbb{R}^+ . For all x in \mathbb{R}^n , let $Q_{s,x}^\theta$ be the unique weakly continuous solution to a martingale problem starting from x at time s .

$(Q_{s,x}^\theta)_{x \in \mathbb{R}^n}$ has the Feller property if for all $t \geq s$, for all f continuous bounded on \mathbb{R}^n , the map $x \rightarrow Q_{s,x}^\theta(f(X_t))$ is continuous. We denote it $T_{st}(f)$.

2. Assume that Θ satisfies hypothesis H_θ . Θ has the Feller property if every $(Q_{s,x}^\theta)$ has the Feller property.

The sets Θ constructed in section 2 have the Feller property:

Lemma 5.5 *The sets Θ given by equation (8) in case of continuous diffusions and by equation (15) in case of diffusions with Levy generator satisfy the property H_θ and have the Feller property.*

Proof The property H_θ has been proved in Proposition 2.5 in case of continuous diffusions. The map X_t is continuous, thus the Feller property is satisfied. In case of diffusions with Levy generator, the property H_θ has been proved in Proposition 2.10. The Feller property has been proved in Proposition 2.9.

Lemma 5.6 *Case of \mathcal{D} or \mathcal{C} . let $Q_{s,x}^\theta$ be the unique weakly continuous solution to a martingale problem starting from x at time s . Assume that $(Q_{s,x}^\theta)_{x \in \mathbb{R}^n}$ has the Feller property. Then*

$$\omega \in \Omega_T^r \rightarrow T_{st}(f)(X_s)(\omega) \quad (59)$$

is the canonical version of the $Q_{r,x}^\theta$ conditional expectation of $f(X_t)$ given \mathcal{B}_s^r .

Proof Let $\psi = f(X_t)$, $t \geq s$. With the notations of Proposition 3.12, $1_{X_s(\omega)=X_s(\omega')}(\psi \circ i)^{\pi_s^r(\omega)}(\omega') = f(X_t)(\omega')$. The Lemma follows then easily from Proposition 3.12, and the definition of $T_{st}(f)$.

Notation 5.7 Let \mathcal{Q} be a weakly relatively compact set of probability measures. Denote as in [12] and [6] $L^1(c)$ the Banach space obtained from the completion and separation of continuous function for the c norm $c(f) = \sup_{Q \in \mathcal{Q}} Q(|f|)$.

Lemma 5.8 Let \mathcal{P} be a set of probability measures on $\mathcal{C}([s, T], \mathbb{R}^n)$. Assume that \mathcal{P} is tight. Let \mathcal{T} be a dense subset of $[s, T]$. The set of continuous bounded coordinates map $f(X_{t_1}, \dots, X_{t_k})$, $s \leq t_1 < \dots < t_k \leq T$, $t_i \in \mathcal{T}$ and $f \in \mathcal{C}_b((\mathbb{R}^n)^k)$ is dense in $L^1(c)$.

Proof Being \mathcal{P} tight, there is a compact subset K of $\Omega^s = \mathcal{C}([s, T], \mathbb{R}^n)$ such that for all $P \in \mathcal{P}$, $P(\omega^s - K) < \epsilon$. Let $\mathcal{E} = \{f(X_{t_1}, \dots, X_{t_k}), s \leq t_1 < \dots < t_k \leq t, t_i \in \mathcal{T}, f \in \mathcal{C}_b((\mathbb{R}^n)^k)\}$. The restriction to K of elements of \mathcal{E} is an algebra which separates the points of the Hausdorff compact space K . Thus from Stone Weierstrass theorem it is dense in $\mathcal{C}(K)$. This proves the result.

Notation 5.9 Let $0 \leq r \leq s \leq T$. For all $j \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$, for all f Borelian on $(\mathbb{R}^n)^{j+k}$, given $r \leq s_1 < \dots < s_j < s \leq t_1 < \dots < t_k \leq T$, denote $f_{(s_1, \dots, s_j), (t_1, \dots, t_k)}$ the map defined on Ω_T^r by $f_{(s_1, \dots, s_j), (t_1, \dots, t_k)}(\omega) = f(X_{s_1}, \dots, X_{s_j}, X_{t_1}, \dots, X_{t_k})(\omega)$.

Proposition 5.10 Case of \mathcal{D} or \mathcal{C} . For all s in \mathbb{R}^+ and $x \in \mathbb{R}^n$, let $Q_{s,x}^\theta$ be the unique weakly continuous solution to a martingale problem starting from x at time s . Assume that for all s $(Q_{s,x}^\theta)_{x \in \mathbb{R}^n}$ has the Feller property. Then for all f continuous bounded (resp. Borelian bounded) on $(\mathbb{R}^n)^{j+k}$, there is a continuous bounded (resp. Borelian bounded) map \hat{f} on $(\mathbb{R}^n)^{j+1}$ such that for all $r < s$ and x in \mathbb{R}^n ,

$$\omega \in \Omega_T^r \rightarrow \hat{f}((X_{s_1}(\omega), \dots, X_{s_j}(\omega), X_s(\omega))) \quad (60)$$

is the canonical version of the $Q_{r,x}$ conditional expectation of $f_{(s_1, \dots, s_j), (t_1, \dots, t_k)}$ given \mathcal{B}_s^r , i.e. $T_s^\theta(f_{(s_1, \dots, s_j), (t_1, \dots, t_k)}) = \hat{f}_{s_1, \dots, s_j, s}$

Proof We prove the proposition for all f continuous bounded on $(\mathbb{R}^n)^{j+k}$. The result follows then for f Borelian bounded from the monotone class theorem.

- Assume that $k = 1$. Let $\psi = f_{(s_1, \dots, s_j), (t_1)}$. With the notations of Prop 3.12, $1_{\{\omega' | \omega'(s) = \omega(s)\}}(\psi \circ i)^{\pi_s^r(\omega)}(\omega') = f_{X_{s_1}(\omega), \dots, X_{s_j}(\omega), X_{t_1}(\omega')}.$

$$\begin{aligned} Q_{s, X_s(\omega)}((\psi \circ i)^{\pi_s^r(\omega)}) &= Q_{s, X_s(\omega)}(1_{\{\omega' | \omega'(s) = \omega(s)\}}(\psi \circ i)^{\pi_s^r(\omega)}) \\ &= Q_{s, X_s(\omega)}(1_{\{\omega' | \omega'(s) = \omega(s)\}}(\phi_\omega(X_{t_1}))) \end{aligned} \quad (61)$$

where ϕ_ω is the continuous function $\phi_\omega(z) = f(X_{s_1}(\omega), \dots, X_{s_j}(\omega), z)$. It follows from the Feller property that for given x_1, \dots, x_j , the map

$$y \rightarrow Q_{s,y}(f(x_1, \dots, x_j, X_{t_1})) \text{ is continuous} \quad (62)$$

We prove now that the map $\hat{f} : (x_1, \dots, x_j, y) \rightarrow Q_{s,y}(f(x_1, \dots, x_j, X_{t_1}))$ is continuous.

The family of probability measures $(Q_{s,y})_{y \in \mathbb{R}^n}$ on Ω_T^s is weakly continuous thus for given $K > 0$, $(Q_{s,y})_{\|y\| \leq K}$ is tight. Thus for all $\epsilon > 0$, there is a compact subset \mathcal{K} of Ω_T^s such that for $\|y\| \leq K$, $Q_{s,y}(\mathcal{K}^c) < \epsilon$. \mathcal{K} is compact for the Skorohod topology, thus from Theorem 12.3 of [2] there is $A > 0$ such that $\sup_{\omega' \in \mathcal{K}} |X_{t_1}(\omega')| \leq A$. Let $K' > 0$. Due to the uniform continuity of f on the compact set $\{\|x_i\| \leq K' \forall 1 \leq i \leq j, \|x_{t_1}\| \leq A\}$, being $Q_{s,y}$ a probability measure, for all $\epsilon > 0$ there is $\eta > 0$ such that for $\|x_i\| \leq K'$, $\|x'_i\| \leq K'$ and $\|x_i - x'_i\| < \eta$,

$$|Q_{s,y}(f(x_1, \dots, x_j, X_{t_1})) - Q_{s,y}(f(x'_1, \dots, x'_j, X_{t_1}))| < 2(1 + \|f\|)\epsilon, \quad \forall y, \quad \|y\| \leq K \quad (63)$$

The continuity of $\hat{f} : (x_1, \dots, x_k, y) \rightarrow Q_{s,y}(g(x_1, \dots, x_j, X_{t_1}))$ follows then easily from (62) and (63).

- It follows from equation (61) and Proposition 3.12 that the map $\omega \rightarrow \hat{f}((X_{s_1}(\omega), \dots, X_{s_j}(\omega), X_s(\omega)))$ is the canonical version of the $Q_{r,x}$ conditional expectation of $f_{(s_1, \dots, s_j), (t_1, \dots, t_k)}$ given \mathcal{B}_s^r .
- By iteration. Assume that the result is satisfied for $k - 1 \geq 1$. We prove it for k . It follows from the first step applied with $s = t_{k-1}$ that there is a continuous function ψ on $\mathbb{R}^{j+(k-1)}$ such that $\psi_{(s_1, \dots, s_j), (t_1, \dots, t_{k-1})}(\omega)$ is the canonical version of the $Q_{r,x}$ conditional expectation of $f_{(s_1, \dots, s_j), (t_1, \dots, t_k)}$ given $\mathcal{B}_{t_{k-1}}^r$, which means that $T_{t_{k-1}}^\theta(f_{(s_1, \dots, s_j), (t_1, \dots, t_k)}) = \psi_{(s_1, \dots, s_j), (t_1, \dots, t_{k-1})}$. The result follows then by iteration making use of the composition rule (42).

□

Notation 5.11 Given $r \leq s$, denote \mathcal{W}_s^r the algebra of continuous coordinates maps \mathcal{B}_s^r measurable.

$$\mathcal{W}_s^r = \{f_{s_1, \dots, s_j} | r \leq s_1 \leq \dots \leq s_j \leq s, j \in \mathbb{N}^*, f \in \mathcal{C}_b(\mathbb{R}^k)\} \quad (64)$$

where

$$f_{s_1, \dots, s_j}(\omega) = f((X_{s_1}(\omega), \dots, X_{s_j}(\omega))) \quad (65)$$

From Proposition 5.10, we deduce the following Corollary:

Corollary 5.12 Assume that Θ satisfied hypothesis H_θ and has the Feller property. For all $Y \in \mathcal{W}_t^r$, $T_s^\theta(Y)$ belongs to \mathcal{W}_s^r .

6 Time Consistent Convex Continuous Dynamic Procedures

In this Section the Polish space is either \mathcal{C}^r or \mathcal{D}^r . We assume that Θ satisfies hypothesis H_θ and has the Feller property. The goal of this Section is to construct time consistent convex procedures having a Feller property, associated to a stable set of probability measures $\mathcal{Q}^\Theta(\Gamma)_{r,y}$ generated by $\Theta(\Gamma)$ (cf Corollary 4.3).

Proposition 6.1 Let $F = \{f_\alpha, \alpha \in A\}$ be a set of l.s.c. functions uniformly bounded from below on $(\mathbb{R}^n)^k$.

- There is a countable subset $\{f_{\alpha_n}, n \in \mathbb{N}\}$ of F such that

$$\sup_{n \in \mathbb{N}} f_{\alpha_n} = \sup_{\alpha \in A} f_\alpha, \quad (66)$$

- If the set F is a lattice upward directed, there is an increasing sequence g_n of elements of F such that

$$\sup_{\alpha \in A} f_\alpha = \lim_{n \rightarrow \infty} g_n, \quad (67)$$

Proof

1. Let $(K_p)_{p \in \mathbb{N}}$ be the closed ball of radius p in $(\mathbb{R}^n)^k$ centered in 0. Let $(\phi_p)_{p \in \mathbb{N}^*}$ be an increasing sequence of continuous functions with compact support contained in K_p such that the restriction of ϕ_p to K_{p-1} is equal to 1.

2. Every function l.s.c. bounded from below is the limit of an increasing sequence of continuous bounded functions uniformly bounded from below. It follows then from 1 that for all α , there is a sequence g_p^α of continuous functions with compact support contained in K_p such that f_α is the increasing limit of g_p^α . Thus

$$\sup_{\alpha \in A} f_\alpha = \sup_{\alpha \in A, p \in \mathbb{N}^*} g_p^\alpha \quad (68)$$

The set of continuous functions with compact support contained in K_p is metrizable separable. It follows that the set $\{g_p^\alpha, \alpha \in A, p \in \mathbb{N}^*\}$ admits a countable dense subset. Thus there is a sequence α_n such that

$$\sup_{\alpha \in A, p \in \mathbb{N}^*} g_p^\alpha = \sup_{n \in \mathbb{N}, p \in \mathbb{N}^*} g_p^{\alpha_n} \quad (69)$$

By construction, for all n, p , $g_p^{\alpha_n} \leq f_{\alpha_n}$. Thus equation (66) follows from (68) and (69).

3. Let f_{α_n} in F be such that equation (66) is satisfied. Define the sequence g_n by iteration: $g_0 = f_{\alpha_0}$ and choose for g_{n+1} a function in F such that $g_{n+1} \geq \sup(g_n, f_{\alpha_{n+1}})$. The sequence g_n is by construction increasing. Let g its limit, $\sup_{\alpha \in A} f_\alpha \geq g \geq \sup_{n \in \mathbb{N}} f_{\alpha_n}$. The result follows from (66). □

We want now to construct general time consistent continuous procedures on \mathcal{C}^r or \mathcal{D}^r . For this we introduce the notion of Feller property for a penalty. Recall that a penalty has been defined in Definition 4.4.

Definition 6.2 *Let $\mathcal{Q} = \mathcal{Q}^\Theta(\Gamma)_{r,y}$ be a stable set of probability measures on $(\Omega^r, \mathcal{B}_T^r)$ generated by $\Theta(\Gamma)$ (cf Corollary 4.3). A penalty $(\alpha_{s,t})_{r \leq s \leq t \leq T}$ defined on \mathcal{Q} is a Feller penalty if*

$$\forall \theta \in \Theta(\Gamma), \exists \beta_{st}^\theta \in \mathcal{C}(\mathbb{R}^n), \alpha_{s,t}(Q_{r,y}^\theta) = \beta_{st}^\theta(X_s) \quad (70)$$

Remark 6.3 *The above Feller property is required only for the probability measures $Q_{r,y}^\theta$ generating the stable set \mathcal{Q} but not for all the probability measures $Q_{r,y}^\gamma$ ($\gamma \in \Theta(\Gamma)_T^r$) in \mathcal{Q} . From the definition of a penalty (Definition 4.4), it follows that the function β_{st}^θ is bounded from above. We do not ask for boundedness.*

Notation 6.4 Let $0 \leq r \leq t$. Recall that \mathcal{W}_t^r (Notation 5.11) is the algebra of continuous coordinates functions. Denote also \mathcal{L}_t the algebra $\mathcal{L}_t = \{f(X_t), f \text{ continuous bounded on } \mathbb{R}^n\}$. In the following

1. \mathcal{H}_t^r denotes either \mathcal{W}_t^r or \mathcal{L}_t .
2. $\hat{\mathcal{H}}_t^r$ denotes either $\hat{\mathcal{W}}_t^r$ or $\hat{\mathcal{L}}_t$, with

$$\hat{\mathcal{W}}_t^r = \{f(X_{t_1}, \dots, X_{t_k}), r \leq t_1 < \dots < t_k \leq t, f \text{ lsc bounded from below}\} \quad (71)$$

$$\hat{\mathcal{L}}_t = \{f(X_t) f \text{ lsc bounded from below}\} \quad (72)$$

The following proposition is an adaptation to this general martingale setting of the ideas used in the proof of Theorem 4.8 of [5].

Proposition 6.5 Let $r \leq s \leq t$. Assume that Θ satisfies hypothesis H_θ and the Feller property. Let Γ be a multivalued Borel mapping. Let α_{st} be a Feller penalty on the stable set of probability measures on $(\Omega^r, \mathcal{B}_T^r)$ generated by $\Theta(\Gamma)$. Let $Y \in \hat{\mathcal{H}}_t^r$,

1. For all $\theta \in \Theta(\Gamma)$, $T_s^\theta(Y) - \alpha_{s,t}(Q_{r,y}^\theta)$ belongs to $\hat{\mathcal{H}}_s^r$
2. For all $\gamma \in \Theta(\Gamma)_t^r$ and $s \in [r, t]$, there is $\eta_s \in \Theta(\Gamma)_t^r$ such that

$$T_s^\gamma(Y) - \alpha_{s,t}(Q_{r,y}^\gamma) \leq T_s^{\eta_s}(Y) - \alpha_{s,t}(P_{r,y}^{\eta_s}) = Z_s \quad (73)$$

and such that $T_s^{\eta_s}(Y) - \alpha_{s,t}(P_{r,y}^{\eta_s}) = Z_s$ belongs to $\hat{\mathcal{H}}_s^r$.

Proof Notice first that for all $\gamma \in \Theta(\Gamma)_t^r$ the map T_s^γ is defined on all functions \mathcal{B}_t^r measurable bounded from below and that if Y, Z are \mathcal{B}_t^r measurable bounded from below, $T_s^\gamma(Y + Z) = T_s^\gamma(Y) + T_s^\gamma(Z)$. This follows from the linearity of T_s^γ on bounded functions and from the continuity from below.

1. Let $\theta \in \Theta(\Gamma)$. Let Y in \mathcal{H}_t^r , it follows from Proposition 5.10 that $T_s^\theta(Y)$ belongs to \mathcal{H}_s^r . If Y belongs to $\hat{\mathcal{H}}_t^r$, Y is the increasing limit of a sequence $Y_n \in \mathcal{H}_t^r$. From the continuity from below of T_s^θ (Corollary 3.14), it follows that $T_s^\theta(Y)$ belongs to $\hat{\mathcal{H}}_s^r$. It follows from the definition of a Feller penalty (Definitions 6.2 and 4.4) that $-\alpha_{st}(Q_{r,y}^\theta)$ belongs to $\hat{\mathcal{H}}_s^r$. This gives the result.

2. $r = s_0 < s_1 < \dots < s_n = t$ be a subdivision associated to γ . For all $u \in [s_{n-1}, s_n[$, $\gamma_u = \sum_{j \in I_{n-1}} 1_{A_{n-1,j}}(\omega) \theta_{n-1,j}(u, X_u(\omega))$. For all $j \in I_{n-1}$, and $s \in [s_{n-1}, s_n[$, from 1., there is an element $Y_{s,j}$ of $\hat{\mathcal{H}}_s^r$ such that $T_s^{\theta_{n-1,j}}(Y) - \alpha_{s,t}(Q_{r,y}^{\theta_{n-1,j}}) = Y_{s,j}$. $Z_s = \sup_{j \in I_{n-1}} Y_{s,j}$ belongs to $\hat{\mathcal{H}}_s^r$. Let $B_{s,j}$ the \mathcal{B}_s^r measurable sets such that $Z_s = \sum_{j \in I_{n-1}} 1_{B_{s,j}} Y_{s,j}$. Let $\eta_s \in \Theta(\Gamma)_t^r$ such that $\eta_s(u)(\omega) = \sum_{j \in I_{n-1}} 1_{B_{s,j}}(\omega) \theta_{n-1,j}(u, X_u(\omega))$ for all $u \in [s, t]$. η_s satisfies the required conditions.
3. For $s \in [s_i, s_{i+1}[$ with $i + 1 < n$, the result is proved downward by induction. $\gamma_s = \sum_{j \in I_i} 1_{A_{i,j}}(\omega) \theta_{i,j}(u, X_u(\omega))$. Assume that Z_v and η_v satisfying equation(73) and $Z_v \in \hat{\mathcal{H}}_s^r$ have been constructed for all $v \geq s_{i+1}$. For all $j \in I_i$, $T_s^{\theta_{i,j}}(Z_{s_{i+1}}) - \alpha_{s,t}(Q_{r,y}^{\theta_{i,j}}) = \phi_{s,j}$ belongs to $\hat{\mathcal{H}}_s^r$. Let $B_{s,j} \in \mathcal{B}_s^r$ such that $Z_s = \sup_{j \in I_i} \phi_{s,j} = \sum_{j \in I_i} 1_{B_{s,j}} \phi_{s,j}$. Let $\eta_s \in \Theta(\Gamma)_t^r$ such that $\eta_s(u) = \eta_{s_{i+1}}(u)$ for all $u \geq s_{i+1}$ and $\eta_s(u)(\omega) = \sum_{j \in I_i} 1_{B_{s,j}} \theta_{i,j}(u, X_u(\omega))$ for all $u \in [s, s_{i+1}[$. η_s and Z_s satisfy the required conditions.

Theorem 6.6 *Let $r \leq s \leq t \leq T$. Let $y \in \mathbb{R}^n$. Assume that Θ satisfies hypothesis H_θ and the Feller property. Let Γ be a multivalued Borel mapping. Let $\mathcal{Q} = \mathcal{Q}^\theta(\Gamma)_{r,y}$ be the stable set of probability measures generated by $\Theta(\Gamma)$. Let (α_{st}) be a Feller penalty on \mathcal{Q} . Denote $T_s^\gamma = T_s^{Q_{r,y}^\gamma}$. For all Y in $\hat{\mathcal{H}}_t^r$, let*

$$\Pi_{s,t}^{r,y}(Y) = \sup_{Q_{r,y}^\gamma \in \mathcal{Q}^\theta(\Gamma)_{r,y}} (T_s^\gamma(Y) - \alpha_{s,t}(Q_{r,y}^\gamma)) \quad (74)$$

1. *For all $Y \in \hat{\mathcal{H}}_t^r$. $\Pi_{s,t}^{r,y}(Y)$ belongs to $\hat{\mathcal{H}}_s^r$. Moreover for all $s \in [r, t]$, there is a sequence $Q_n = Q_{r,y}^{\gamma_n}$ in \mathcal{Q} , γ_n in $\Theta(\Gamma)_t^r$ such that $T_s^{\gamma_n}(Y) - \alpha_{s,t}(Q_{r,y}^{\gamma_n})$ belongs to $\hat{\mathcal{H}}_s^r$, and such that $\Pi_{s,t}^{r,y}(Y)$ is the increasing limit:*

$$\Pi_{s,t}^{r,y}(Y) = \lim [T_s^{\gamma_n}(Y) - \alpha_{s,t}(Q_{r,y}^{\gamma_n})] \quad (75)$$

2. *$\Pi_{s,t}^{r,y}$ is a convex monotone map continuous from below on $\hat{\mathcal{H}}_t^r$ with values in $\hat{\mathcal{H}}_s^r$.*

3. *For all $r \leq s \leq t \leq u$, for all $Y \in \hat{\mathcal{H}}_u^r$,*

$$\Pi_{s,u}^{r,y}(Y) = \Pi_{s,t}^{r,y}(\Pi_{t,u}^{r,y}(Y)) \quad (76)$$

4. In case $\mathcal{H}_t^r = \mathcal{L}_t$, the sequence γ_n of 1. can be choosen independently of r .

Proof

1. Let $Y \in \hat{\mathcal{H}}_t^r$. For $Q_{r,y}^\gamma \in \mathcal{Q}$, let $Y_s^\gamma = T_s^\gamma(Y) - \alpha_{s,t}(Q_{r,y}^\gamma)$. Let $\mathcal{Z} = \{Y_s^\gamma \mid Q_{r,y}^\gamma \in \mathcal{Q}, \gamma \in \Theta(\Gamma)_t^r \mid Y_s^\gamma \in \hat{\mathcal{H}}_s^r\}$. It follows from the stability by bifurcation of \mathcal{Q} , from the expression of T^γ (equation (46)) and from the locality of the penalty that \mathcal{Z} is a lattice upward directed.
 - If $\mathcal{H}_t^r = \mathcal{W}_t^r$, there is a lsc function f bounded from below such that $Y = f(X_{s_1}, X_{s_2}, \dots, X_{s_l})$. Let j such that $s_j < s \leq s_{j+1}$. From Proposition 5.10 for all $\theta \in \Theta(\Gamma)$, $Y_s^\theta = g(X_{s_1}, \dots, X_{s_j}, X_s)$ for some function g continuous bounded from below on $(\mathbb{R}^n)^{j+1}$. It follows from the proof of Proposition 6.5 that $\{g \mid g(X_{s_1}, \dots, X_{s_j}, X_s) \in \mathcal{Z}\}$ is a lattice of lsc functions bounded from below on $(\mathbb{R}^n)^{j+1}$. The result follows from Proposition 6.1.
 - In case $\mathcal{H}_t^r = \mathcal{L}_t$, $Y = f(X_t)$. For all $\gamma \in \Theta(\Gamma)_T^r$, $Y_s^\gamma = g(X_s)$ for some function g Borelian bounded from below on $(\mathbb{R}^n)^{j+1}$ independent of r . A similar proof as the above one proves 1. and 4.
2. From the linearity of T_s^γ on bounded functions and the continuity from below, it follows that for all non negative real numbers c and d , for all Y, Z in \mathcal{H}_t^r , $T_s^\gamma(cY + dZ) = cT_s^\gamma(Y) + dT_s^\gamma(Z)$. The convexity of $\Pi_{s,t}^{r,y}$ follows. Let Y_n be increasing to Y . Every T_s^γ is continuous from below thus $\Pi_{s,t}^{r,y}(Y) = \sup_\gamma T_s^\gamma(\lim Y_n) = \sup_{\gamma,n} T_s^\gamma(Y_n) = \sup_n \Pi_{s,t}^{r,y}(Y_n)$.
3. Let $Y \in \hat{\mathcal{H}}_u^r$. From 1., $\Pi_{t,u}^{r,y}(Y)$ belongs to $\hat{\mathcal{H}}_t^r$ and we have the following increasing limits:

$$\Pi_{s,t}^{r,y}(\Pi_{t,u}^{r,y}(Y)) = \lim_{n \rightarrow \infty} [T_s^{\gamma_n}(\Pi_{t,u}^{r,y}(Y)) - \alpha_{s,t}(Q_{r,y}^{\gamma_n})] \quad (77)$$

$$\Pi_{t,u}^{r,y}(Y) = \lim_{k \rightarrow \infty} [T_t^{\delta_k}((Y)) - \alpha_{t,u}(Q_{r,y}^{\delta_k})] \quad (78)$$

Fom the continuity from below of $T_s^{\gamma_n}$, it follows that

$$\Pi_{s,t}^{r,y}(\Pi_{t,u}^{r,y}(Y)) = \sup_{n,k} (T_s^{\gamma_n}[T_t^{\delta_k}((Y)) - \alpha_{t,u}(Q_{r,y}^{\delta_k})] - \alpha_{s,t}(Q_{r,y}^{\gamma_n})) \quad (79)$$

From the stability of \mathcal{Q} , for all n and k , let $\lambda_{n,k}$ be defined as in Theorem 4.2 i), such that $T_s^{\gamma_n}(\phi) = T_s^{\lambda_{n,k}}(\phi)$ for all $\phi \in \mathcal{B}_t^r$ measurable

bounded and $T_t^{\delta_k} = T_t^{\lambda_{n,k}}$. From the chain rule, $T_s^{\gamma_n} \circ [T_t^{\delta_k}] = T_s^{\lambda_{n,k}}$. From the definition of $\lambda_{n,k}$, the local property of the penalty and the cocycle condition, it follows that $T_s^{\gamma_n}[-\alpha_{t,u}(Q_{r,y}^{\delta_k})] - \alpha_{s,t}(Q_{r,y}^{\gamma_n}) = -\alpha_{s,u}(Q_{r,y}^{\lambda_{n,k}})$. Thus from (79), it follows that

$$\Pi_{s,t}^{r,y}(\Pi_{t,u}^{r,y}(Y)) = \sup_{n,k} [T_s^{\lambda_{n,k}}(Y) - \alpha_{s,u}(Q_{r,y}^{\lambda_{n,k}})] \leq \Pi_{s,u}^{r,y}(Y) \quad (80)$$

Conversely, $\Pi_{s,u}^{r,y}(Y)$ is the increasing limit of $T_s^{\nu_j}(Y) - \alpha_{s,u}(Q_{r,y}^{\nu_j})$. Making use of the chain rule for $T_s^{\nu_j}$ and of the cocycle condition, we get the inequality $\Pi_{s,u}^{r,y}(Y) \leq \Pi_{s,t}^{r,y}(\Pi_{t,u}^{r,y}(Y))$.

4. If $Y \in \mathcal{L}_t$, $Y = f(X_t)$.

Definition 6.7 Let $r \leq s \leq t \leq T$. Let Θ satisfying hypothesis H_θ . let $\mathcal{Q} = \mathcal{Q}^\theta(\Gamma)_{r,y}$ be the stable set of probability measures generated by $\Theta(\Gamma)$. Let (α_{st}) be a Feller penalty on \mathcal{Q} . For all Y in $\hat{\mathcal{H}}_t^r$, let

$$\Pi_{s,t}^{r,y}(Y) = \sup_{Q_{r,y}^\gamma \in \mathcal{Q}^\theta(\Gamma)_{r,y}} (T_s^{Q_{r,y}^\gamma}(Y) - \alpha_{s,t}(Q_{r,y}^\gamma)) \quad (81)$$

One says that $\Pi_{s,t}^{r,y}$ is a time consistent dynamic procedure on $\hat{\mathcal{H}}_t^r$ if all properties of Theorem 6.6 are satisfied.

Proposition 6.8 Let $\mathcal{P} = \mathcal{Q}^\theta(\Gamma)_{r,y}$. Denote $\overline{\mathcal{W}}_t^r$ the closure of the lattice vector space \mathcal{W}_t^r for the norm $\sup_{Q \in \mathcal{P}} E_Q(|X|)$. Assume that for all s , and $X \in \mathcal{W}_t^r$, $\Pi_{s,t}^{r,y}(X)$ belongs to $\overline{\mathcal{W}}_s^r$. The procedure $\Pi_{s,t}^{r,y}$ can be uniquely extended to $\overline{\mathcal{W}}_t^r$. The time consistency (equation 76) extends to $\overline{\mathcal{W}}_t^r$. In case where \mathcal{P} is a weakly relatively compact set of probability measures on $\mathcal{C}([r, T], \mathbb{R}^n)$, $\overline{\mathcal{W}}_t^r$ is equal to $L^1(c)$.

Proof Let X, Y in \mathcal{W}_t^r . From equation (81), $\Pi_{s,t}^{r,y}(Y)$ is the increasing limit of $T_s^{\gamma_n}(Y) - \alpha_{s,t}(Q_{r,y}^{\gamma_n})$. It follows that $\Pi_{st}^{r,y}(Y) \leq \Pi_{st}^{r,y}(X) + \sup_{n \in \mathbb{N}} T_s^{\gamma_n}(Y - X)$. Changing the roles of X and Y , it follows that

$$|\Pi_{st}^{r,y}(Y) - \Pi_{st}^{r,y}(X)| \leq \sup_{n,k} [T_s^{\gamma_n}(|Y - X|), T_s^{\delta_k}(|Y - X|)] \quad (82)$$

As in the proof of equation (75), it follows from Proposition 6.1 that $\sup_{Q^\gamma \in \mathcal{P}} T_s^\gamma(|X - Y|)$ is the increasing limit of a sequence $T_s^{\alpha_n}(|X - Y|)$. It follows from equation (82) and the stability of \mathcal{P} that

$$\sup_{P \in \mathcal{P}} E_P(|\Pi_{st}^{r,y}(Y) - \Pi_{st}^{r,y}(X)|) \leq \sup_{P \in \mathcal{P}} E_P(|X - Y|) \quad (83)$$

It follows from equation (83), that $\Pi_{st}^{r,y}$ can be uniquely extended to $\overline{\mathcal{W}}_t^r$, with values in $\overline{\mathcal{W}}_s^r$, equation (76) is then satisfied on $\overline{\mathcal{W}}_t^r$.

The last result follows from Lemma 5.8.

7 Construction of Feller penalties

7.1 Penalties for continuous diffusions

In this Section we restrict to the case of continuous paths. We consider the martingale problem for diffusions. In [5] we have constructed Feller penalties associated to probability measures solution to the martingale problem for continuous diffusions. In [5] the probability measures in the stable set were all solutions to a martingale problem with the same function $a(t, x)$. In the present paper the probability measures are no more equivalent, they are even not dominated. The construction of the penalties that we describe below is the generalization of the construction that we have done in [5]. The ideas are the same. In this paper we chose the parameters a and b .

Definition 7.1 Θ is the set of continuous bounded functions $\theta(t, x) = (a(t, x), b(t, x))$ with values in $M_n(\mathbb{R}) \times \mathbb{R}^n$ such that for all (t, x) the matrix $a(t, x)$ is invertible. Let Γ be a multivalued Borel mapping from $\mathbb{R}_+ \times \mathbb{R}^n$ to $M_n(\mathbb{R}) \times \mathbb{R}^n$, $\Theta(\Gamma) = \{\theta \in \Theta, \Gamma \text{ valued}\}$.

For general $\gamma = (\eta, \mu) \in \Theta_T^r$ we denote $P_{r,y}^{\eta,\mu}$ the unique solution to the martingale problem starting from y at time r constructed previously. In the case $\gamma \in \Theta$, $\gamma(u, \omega) = \theta(u, X_u(\omega))$ for some continuous $\theta(u, x)$. When we want to emphasize the fact that we refer to continuous parameters η and μ , the probability measure $P_{r,y}^{\eta,\mu}$ will be denoted $Q_{r,y}^{a,b}$.

As in [5] we introduce hypothesis H_g .

Definition 7.2 Hypothesis H_g

1. $g : \mathbb{R}_+ \times \mathbb{R}^n \times (M_n(\mathbb{R}) \times \mathbb{R}^n) \rightarrow \mathbb{R} \cup \infty$ is a “Caratheodory function on Γ ”

More precisely, g is Borelian and for all u , the restriction of g_u to $\{(x, y), y \in \Gamma(u, x)\}$ is continuous ($g_u(x, y) = g(u, x, y)$).

2. g has polynomial growth on Γ

$$\forall (t, x, y), y \in \Gamma(t, x), |g(t, x, y)| \leq C(1 + \|x\|^m) \quad (84)$$

3. g is bounded from above on Γ which means that g is bounded from above on $\{(t, x, y), y \in \Gamma(t, x)\}$

This last condition on Γ which was not present in [5] could be suppressed with a little more work.

Recall that from Corollary 4.3, the set $\mathcal{Q}^\Theta(\Gamma)_{r,y} = \{Q_{r,y}^\gamma, \gamma \in \Theta(\Gamma)_T^r\}$ is stable.

Proposition 7.3 *Let Γ be a multivalued Borel mapping. Let $\Theta(\Gamma)$ be as in Definition 7.1. Assume that g satisfies Hypothesis H_g . Let $t > 0$.*

1. *For all $\theta = (a, b) \in \Theta(\Gamma)$, there is a real valued continuous map $(s, x) \in [0, t] \times \mathbb{R}^n \rightarrow L_t^{a,b}(g)(s, x)$ such that*

$$Q_{s,y}^{a,b}(\int_s^t g(u, X_u, \theta(u, X_u))du) = L_t^{a,b}(g)(s, y) \quad \forall s \in [0, t] \text{ and } y \in \mathbb{R}^n \quad (85)$$

2. *For all $0 \leq r \leq s \leq t$ and all $y \in \mathbb{R}^n$, for $Q = Q_{r,y}^{a,b}$,*

$$T_s^Q[-\int_s^t g(u, X_u, \theta(u, X_u))du] = -L_t^{a,b}(g)(s, X_s(\omega)) \quad (86)$$

Notice that for all $y \in \mathbb{R}^n$ $L_t^{a,b}(g)(t, x) = 0$

Proof 1. is an application of the first statement of Proposition 4.2 of [5]

2. $-g$ is bounded from below, so making use of the extension of T_s^Q defined on Corollary 3.14 it follows that

$$\begin{aligned} T_s^Q[-\int_s^t g(u, X_u, \theta(u, X_u))du] &= Q_{s, X_s(\omega)}^{a,b}[-\int_s^t g(u, X_u, \theta(u, X_u))du] \\ &= -L_t^{a,b}(g)(s, X_s(\omega)) \end{aligned} \quad (87)$$

We define now the penalty of every probability measure $Q_{r,y}^\gamma$ in $\mathcal{Q}^\Theta(\Gamma)_{r,y}$.

Definition 7.4 *For all $Q_{r,y}^\gamma$, for all $r \leq s \leq t \leq T$, we define the penalty*

$$\alpha_{s,t}(Q_{r,y}^\gamma) = -T_s^\gamma[-\int_s^t g(u, X_u(\omega), \gamma(u, \omega))du] \quad (88)$$

Corollary 7.5 *The above definition provides a Feller penalty according to Definitions 4.4 and 6.2.*

Proof It follows from hypothesis H_g that for all $r \leq s \leq t$, the function $-\int_s^t g(u, X_u(\omega), \gamma(u, \omega)) du$ is \mathcal{B}_t^r measurable bounded from below, thus from Corollary 3.14, $\alpha_{s,t}(Q_{r,y}^\gamma)$ is \mathcal{B}_s^r measurable bounded from above. The cocycle condition for $\alpha_{s,t}(Q_{r,y}^\gamma)$ follows from the chain rule for T^γ (Corollary 3.14) and the additivity of T_s^γ on functions bounded from below. The local condition follows from the Definition of the penalty. The Feller property of the penalty follows from Proposition 7.3.

7.2 Penalties for diffusions with Levy generators

In this Section we consider probability measures on the Polish space \mathcal{D}^r of càdlàg paths endowed with the Skorhokod topology. The notations are those of Section 2.2.

Definition 7.6 1. Let $C > 0$. \mathcal{M}_C denotes the set of σ -finite measures μ on $\mathbb{R}^n - \{0\}$ such that

$$\int [||y||^2 1_{||y|| \leq 1} + ||y|| 1_{||y|| > 1}] \mu(dy) \leq C \quad (89)$$

2. Θ is the set of (a, b, M) such that

- a, b are continuous bounded functions on $\mathbb{R}_+ \times \mathbb{R}^n$, with values in $M^n(\mathbb{R})$, \mathbb{R}^n , and for all (t, x) $a(t, x)$ is invertible.
- M satisfies hypothesis M_C (Definition 2.6), i.e. for all (t, x) , $M(t, x) \in \mathcal{M}_C$. Furthermore for all Borelian subset Δ of $\mathbb{R}^n - \{0\}$,

$$\int_{\Delta} \frac{||y||^2}{1 + ||y||^2} M(s, x, dy) \quad (90)$$

is a continuous bounded function of (s, x) .

3. \mathcal{M}_C is endowed with the weak topology. Let Γ be a closed convex multivalued Borel mapping from $\mathbb{R}_+ \times \mathbb{R}^n$ to $M_n(\mathbb{R}) \times \mathbb{R}^n \times \mathcal{M}_C$. Let $\Theta(\Gamma) = \{\theta \in \Theta, \Gamma \text{ valued}\}$.

We want now to define the penalty of every probability measure $Q_{r,y}^\gamma$ in $\mathcal{Q}^\Theta(\Gamma)_{r,y}$. Recall that Θ_T^r is described by Definition 3.1. For all $\gamma \in \Theta(\Gamma)_T^r$, write $\gamma(u, \omega) = (\eta, \lambda, \mu)(u, \omega)$, where η takes values in $M_n(\mathbb{R})$, λ in \mathbb{R}^n and μ takes values in \mathcal{M}_C .

Definition 7.7 Let g be a real valued function on $\mathbb{R}^+ \times M_n(\mathbb{R}) \times \mathbb{R}^n \times \mathcal{M}_C$ such that for all $\theta = (a, b, M) \in \Theta(\Gamma)$, $g^\theta(u, x) = g(u, x, a(u, x), b(u, x), M(u, x))$ is continuous bounded. We define the penalty

$$\alpha_{s,t}(Q_{r,y}^\gamma) = T_s^\gamma \left[\int_s^t g(u, X_u(\omega), \eta(u, \omega), \lambda(u, \omega), \mu(u, \omega)) du \right] \quad (91)$$

Proposition 7.8 The above definition provides a penalty according to Definition 4.4. The penalty is a Feller penalty (Definition 6.2). More precisely, for all $\theta = (a, b, M) \in \Theta(\Gamma)$, there is a continuous function $L_t^\theta(g)$ on $[0, t] \times \mathbb{R}^n$ such that $\forall s \in [0, t]$ and $y \in \mathbb{R}^n$,

$$Q_{s,y}^{a,b,M} \left(\int_s^t g(u, X_u(\omega), a(u, X_u(\omega)), b(u, X_u(\omega)), M(u, X_u(\omega))) du \right) = L_t^\theta(g)(s, y) \quad (92)$$

For all $0 \leq r \leq s \leq t$ and all $y \in \mathbb{R}^n$, for $Q = Q_{r,y}^{a,b,M}$,

$$T_s^Q \left[\int_s^t g(u, X_u(\omega), a(u, X_u(\omega)), b(u, X_u(\omega)), M(u, X_u(\omega))) du \right] = L_t^\theta(g)(s, X_s(\omega)) \quad (93)$$

Proof

1. It follows easily from the hypothesis, and the description of $\Theta(\Gamma)_T^r$ (cf Definition 3.1) that for all $\gamma = (\eta, \lambda, \mu) \in \Theta(\Gamma)_T^r$, the function $\int_s^t g(u, X_u(\omega), \eta(u, \omega), \lambda(u, \omega), \mu(u, \omega))$ is \mathcal{B}_t^r -measurable bounded. From Proposition 3.13 it follows that $\alpha_{s,t}(Q_{r,y}^\gamma)$ is \mathcal{B}_s^r measurable bounded.
2. The cocycle condition follows easily from the definition of the penalty (91), the linearity for T^γ and the chain rule for T^γ .
3. locality Let $\gamma, \eta \in \Theta_T^r$, for all $A \in \mathcal{B}_s^r$, if $1_A \gamma(u, \omega) = 1_A \eta(u, \omega)$, for all $u \in]s, t]$. from Proposition 3.13 equation (46), T_s^γ and T_s^η coincide on bounded variables \mathcal{B}_t^r measurable. The locality follows then from the definition of the penalty.
4. Let $\theta \in \Theta(\Gamma)$, $\theta = (a, b, M)$, a, b are continuous bounded functions of (u, x) , and M satisfies hypothesis M_C . Let $\epsilon > 0$. Let K such

that equation (10) is satisfied for all $0 \leq r \leq t$ and $\|y\| \leq D$. The function g^θ is bounded and uniformly continuous on $[0, T] \times \{\|x\| \leq K\}$. Making use of equation (11) and of arguments similar to those of Proposition 2.9, it follows that for all $s' \leq t$, there are $s' \leq t_1 < t_2 < \dots < t_k \leq t$ and a continuous bounded function $f_{s',t}$ on $(\mathbb{R}^n)^k$ such that for all $\|y\| \leq D$, for all $0 \leq r \leq s'$,

$$Q_{r,y}^{a,b,M}(|\int_{s'}^t g^\theta(u, X_u(\omega))du - f_{s',t}(X_{t_1}, X_{t_2}, \dots, X_{t_k})|) < \epsilon \quad (94)$$

From Proposition 2.9, there is a continuous function $\hat{f}_{s',t}$ on $[0, s'] \times \mathbb{R}^n$ such that $Q_{r,y}^{a,b,M}(f_{s',t}(X_{t_1}, X_{t_2}, \dots, X_{t_k})) = \hat{f}_{s',t}(r, y)$. It follows from (94) that $Q_{r,y}^{a,b,M}(\int_s^t g^\theta(u, X_u(\omega))du)$ is the uniform limit on $[0, s'] \times \{\|y\| \leq D\}$ of a sequence of continuous functions. It is thus continuous. The function g being bounded, the continuity on $[0, t] \times \mathbb{R}^n$ of the function L_t^θ defined by equation (92) follows then easily. Equation (93) follows from Proposition 3.12.

8 Feller property of the time consistent dynamic procedure

In all the following we restrict to the case of continuous diffusions or diffusions with Levy generator. The notations and hypothesis are those of Section 7. We prove now that with the specific choice for the penalty which was made in Subsections 7.1 and 7.2, the procedures $\Pi_{s,t}^{r,y}$ that we have constructed for all given r and y are strongly connected and have a Feller property. We introduce now a larger set of parameters: $\tilde{\Theta}(\Gamma)_T^r$. The definition of $\tilde{\Theta}(\Gamma)_T^r$ is similar to the definition of $\Theta(\Gamma)_T^r$ (cf Definition 3.1), but now the set I_0 is a finite set and not necessarily a singleton. Every element $\tilde{\gamma}$ of $\tilde{\Theta}(\Gamma)_T^r$, satisfies equation (16). For $\tilde{\gamma} \in \tilde{\Theta}(\Gamma)_T^r$, the process $(Z_{r,u}^{\tilde{\gamma}})_{r \leq u \leq T}$ is defined by equation (17).

Remark 8.1 *Let $0 \leq s < t$. Let $x \in \mathbb{R}^n$. Let $\tilde{\gamma} \in \tilde{\Theta}(\Gamma)_t^s$. The sets $A_{0,j}, j \in I_0$ belong to the σ -algebra generated by X_s and form a partition of Ω_t^r . It follows that there is $j_x \in I_0$ such that $\{X_s(\omega) = x\} \subset A_{0,j_x}$. Therefore a probability measure Q on $(\Omega^s, \mathcal{B}_t^s)$ is solution to the martingale problem $(Z_{s,u}^{\tilde{\gamma}})_{s \leq u \leq t}$ starting from x at time s (Definition 2.3) if and only if Q is solution to the martingale problem $(Z_{s,u}^{\gamma_x})_{s \leq u \leq t}$ starting from x at time*

s , where $(\gamma_x)(u, \omega) = \theta_{0,j_x}(u, X_u(\omega))$ for all $s \leq u < s_1$, and $(\gamma_x)(u, \omega) = \tilde{\gamma}(u, \omega)$ for all $s_1 \leq u$. Thus γ_x belongs to $\Theta(\Gamma)_t^s$.

The following proposition is a more precise version of Proposition 6.5.

Proposition 8.2 *Let h be a continuous function on \mathbb{R}^n bounded from below. For all $\gamma \in \Theta(\Gamma)_t^r$ there is a continuous function h^γ on $[0, t] \times \mathbb{R}^n$ bounded from below and for all $r \leq s \leq t$, there is $\eta_s \in \tilde{\Theta}(\Gamma)_t^s$ such that*

1.

$$\forall x \in \mathbb{R}^n, T_s^{P_{s,x}^{\eta_s}}(h(X_t)) - \alpha_{s,t}(P_{s,x}^{\eta_s}) = h^\gamma(s, x) \quad (95)$$

There is a partition B_j of \mathbb{R}^n in Borelian sets, and $\eta_s^j \in \Theta(\Gamma)_t^s$ such that

$$\forall x \in B_j, T_s^{P_{s,x}^{\eta_s^j}}(h(X_t)) - \alpha_{s,t}(P_{s,x}^{\eta_s^j}) = h^\gamma(s, x) \quad (96)$$

$$\forall x \in B_j, h^{\eta_s^j}(u, x) = h^\gamma(u, x), \forall s \leq u \leq t \quad (97)$$

2. For all $\eta \in \Theta(\Gamma)_t^r$ whose restriction to $[s, t]$ is equal to η_s ,

$$\begin{aligned} \forall y \in \mathbb{R}^n, T_s^{P_{r,y}^\eta}(h(X_t)) - \alpha_{s,t}(P_{r,y}^\eta) &= h^\gamma(s, X_s) \\ T_s^{Q_{r,y}^\gamma}(h(X_t)) - \alpha_{s,t}(Q_{r,y}^\gamma) &\leq T_s^{P_{r,y}^\eta}(h(X_t)) - \alpha_{s,t}(P_{r,y}^\eta) \\ h^\gamma(u, x) &= h^\eta(u, x) \quad \forall s \leq u \leq t \end{aligned} \quad (98)$$

Proof

- Step 1: Let $\theta \in \Theta(\Gamma)$, let h^θ be the function defined on $[0, t] \times \mathbb{R}^n$ by

$$T_s^{P_{s,x}^\theta}(h(X_t)) - \alpha_{s,t}(P_{s,x}^\theta) = h^\theta(s, x) \quad (99)$$

The continuity of the map h^θ on $[0, t] \times \mathbb{R}^n$ follows from Theorem 7.1 of [33] (see Proposition 2.7 of [5] for details), and from Proposition 7.3 in case of diffusions. In case of diffusions with Levy generator, the continuity of h^θ follows from Propositions 2.9 and 7.8. Equation

$$\forall 0 \leq r \leq s \leq t, \forall y \in \mathbb{R}^n, T_s^{P_{r,y}^\theta}(h(X_t)) - \alpha_{s,t}(P_{r,y}^\theta) = h^\theta(s, X_s) \quad (100)$$

follows then from Lemma 5.6, Propositions 7.3 and 7.8.

- Step 2: The proof of the existence of h^γ and η_s satisfying (98) follows the proof of Theorem 4.8 in [5]. We sketch the proof. Let $r = s_0 < s_1 < \dots < s_n = t$ be a subdivision associated to γ . For $u \in [s_i, s_{i+1}[$, $\gamma(u, \omega) = \sum_{j \in I_i} 1_{A_{i,j}}(\omega) \theta_{i,j}(u, X_u(\omega))$, $\theta_{i,j} \in \Theta(\Gamma)$. h^γ and η_s are defined recursively. Assume that h^γ is defined and continuous on $[s_{i+1}, t] \times \mathbb{R}^n$, and that η_u is defined for $u \in [s_{i+1}, t]$. Let $h_{i,j}$ be the continuous function on $[s_i, s_{i+1}]$ such that $\forall s \in [s_i, s_{i+1}]$,

$$\forall x \in \mathbb{R}^n, T_s^{P_{s,x}^{\theta_{i,j}}}(h^\gamma(s_{i+1}, X_{s_{i+1}})) - \alpha_{s,s_{i+1}}(P_{s,x}^{\theta_{i,j}}) = h_{i,j}(s, x) \quad (101)$$

For $s_i \leq s < s_{i+1}$, let $h^\gamma(s, x) = \sup_{j \in I_i} h_{i,j}(s, x)$. Given $s \in [s_i, s_{i+1}[$, there is a Borelian partition $B_j, j \in I_i$ of \mathbb{R}^n such that $h^\gamma(s, x) = \sum_{j \in I_i} 1_{B_j}(x) h_{i,j}(s, x)$. Let $\eta_s(u, \omega) = \eta_s^j(u, \omega) = \eta_{s_{i+1}}(u, \omega)$ for $u \geq s_{i+1}$. Let $\eta_s^j(u, \omega) = \theta_{i,j}(u, X_u(\omega))$ and $\eta_s(u, \omega) = \sum_j 1_{B_j}(X_s) \theta_{i,j}(u, X_u(\omega))$ for $s \leq u < s_{i+1}$. We end the proof of 1. making use of the expression of $T_s^{Q_{r,y}^\gamma}$, cf Theorem 3.13 and of Remark 8.1. Notice that η_r belongs to $\Theta(\Gamma)_t^r$, $\eta_s^j \in \Theta(\Gamma)_t^s$ and $\eta_s \in \tilde{\Theta}(\Gamma)_t^s$

- Step 3: It follows from Step 1, from the expressions of $T_s^{Q_{r,y}^\gamma}$, $T_s^{P_{r,y}^\eta}$, and the properties of the penalty that the construction given above leads to h^γ and η_s such that equation (98) is satisfied for all $\eta \in \theta(\Gamma)_t^r$ whose restriction to $[s, t]$ is equal to η_s .

□

Theorem 8.3 1. Let $0 \leq r \leq T$. Let $\hat{\mathcal{W}}_T^r$ be the cone of coordinate functions $f(X_{t_1}, \dots, X_{t_k})$, f lower semi continuous bounded from below (Notation 6.4). Let $(\Pi_{s,t}^{r,y})_{r \leq s \leq t \leq T}$ be defined on $\hat{\mathcal{W}}_T^r$ by

$$\Pi_{s,t}^{r,y}(Y) = \sup_{Q_{r,y}^\gamma \in \mathcal{Q}^\theta(\Gamma)_{r,y}} (T_s^{Q_{r,y}^\gamma}(Y) - \alpha_{s,t}(Q_{r,y}^\gamma)) \quad (102)$$

where $\alpha_{s,t}$ is the penalty constructed in Section 7. Then $(\Pi_{s,t}^{r,y})_{r \leq s \leq t \leq T}$ is a time consistent convex dynamic procedure on $\hat{\mathcal{W}}_t^r$.

2. It has the following Feller property: For all h lower semi continuous function on \mathbb{R}^n bounded from below, there is a lower semi-continuous function \tilde{h} bounded from below on $[0, t] \times \mathbb{R}^n$ such that

$$\forall x \in \mathbb{R}^n, \Pi_{s,t}^{s,x}(h(X_t)) = \tilde{h}(s, x) \quad (103)$$

$$\forall 0 \leq r < s \leq t, \forall y \in \mathbb{R}^n, \Pi_{s,t}^{r,y}(h(X_t)) = \tilde{h}(s, X_s) \quad (104)$$

Furthermore for all given r and y , for all $r < s \leq t$, there is a sequence $\eta_n \in \Theta(\Gamma)_t^r$ such that $\Pi_{s,t}^{r,y}(h(X_t))$ is the increasing limit of $(T_s^{P_{r,y}^{\eta_n}}(h(X_t)) - \alpha_{s,t}(P_{r,y}^{\eta_n}))$

Proof It follows from Corollary 7.5 and Proposition 7.8 that α is a Feller penalty. Thus from Theorem 6.6, and Definition 6.7, $\Pi_{s,t}^{r,y}$ is a time consistent convex dynamic procedure on $\hat{\mathcal{W}}_t^r$.

With the notations of Proposition 8.2, let $\tilde{h} = \sup_{\gamma \in \Theta(\Gamma)_t^0} h^\gamma$. From equation (97), it follows that for all s and all $0 \leq r \leq s$,

$$\sup_{\gamma \in \Theta(\Gamma)_t^r} h^\gamma(s, x) = \sup_{\eta \in \Theta(\Gamma)_t^s} h^\eta(s, x) \quad (105)$$

It follows from Proposition 8.2 and equation (105) that $\forall r \leq s \leq t$,

$$\begin{aligned} \Pi_{s,t}^{r,y}(h(X_t)) &= \sup_{Q_{r,y}^\gamma \in \mathcal{Q}^\theta(\Gamma)_{r,y}} (T_s^{Q_{r,y}^\gamma}(h(X_t)) - \alpha_{s,t}(Q_{r,y}^\gamma)) \\ &= \sup_{\eta \in \Theta(\Gamma)_t^r} h^\eta(s, X_s) = \tilde{h}(s, X_s) \end{aligned} \quad (106)$$

The last assertion follows from Theorem 6.6.

9 Viscosity solution

The pocedures are the procedures constructed in the previous Section in case of Diffusions with Levy generator or in case of continuous diffusions. From now on, one assumes furthermore that Γ is a multivalued Borel mapping convex and closed valued.

9.1 Viscosity supersolution

One assumes that for all K large enough, Γ_K is lower hemicontinuous (cf Definition 16.2 in [1], where $\Gamma_K(t, x) = \{(a, b, \mu) \in \Gamma(t, x), \|a\| \leq K, \|b\| \leq K, \int_{\mathbb{R}^n - \{0\}} [\|y\|^2 1_{\|y\| \leq 1} + \|y\| 1_{\|y\| > 1}] d\mu(y) \leq K\}$ in case of diffusions with Levy generator (and $\Gamma_K(t, x) = \{(a, b) \in \Gamma(t, x), \|a\| \leq K, \|b\| \leq K$ in case of continuous diffusions). Let $(t_0, x_0) \in [0, t[\times \mathbb{R}^n$. Let $\phi \in \mathcal{C}_b^{1,2}([0, t] \times \mathbb{R}^n)$ such that

$$0 = v(t_0, x_0) - \phi(t_0, x_0) = \min(v(t, x) - \phi(t, x)) \quad (107)$$

Lemma 9.1 *Let $\phi \in \mathcal{C}_b^{1,2}([0, t] \times \mathbb{R}^n)$. Assume that M satisfies the hypothesis M_C of Definition 7.6. Let*

$$\tilde{K}\phi(s, x)(y) = \phi(s, x + y) - \phi(s, x) - \frac{y^* \nabla \phi(s, x)}{1 + \|y\|^2} \quad (108)$$

The function $K\phi(s, x) = \int_{\mathbb{R}^n - \{0\}} \tilde{K}\phi(s, x)(y) M(s, x, dy)$ is a continuous function of (s, x) .

Proof The functions $\tilde{K}\phi(s, x)$ given by (108) are uniformly bounded. Thus for all $\epsilon > 0$, there is $K > 0$ such that for all $\mu \in \mathcal{M}_C$ (Definition 2.6),

$$\forall (s, x), \int_{\|y\| \geq K} \tilde{K}\phi(s, x)(y) \mu(dy) < \epsilon \quad (109)$$

It follows from the Taylor formula with integral remainder that $\tilde{K}\phi(s, x)(y) - \tilde{K}\phi(s_0, x_0)(y) = y^* (\nabla(\phi)(s, x) - \nabla(\phi)(s_0, x_0)) \frac{\|y\|^2}{1 + \|y\|^2} + \int_0^1 \text{Tr}(D^2\phi(s, x + uy) - D^2\phi(s_0, x_0 + uy)) yy^* (1 - u) du$. Let $K_0 > 0$. Making use of the uniform continuity of $(s, x) \rightarrow \nabla(f)(s, x)$ and $(s, x) \rightarrow D^2\phi(s, x)$ on compact spaces it follows that for $|s - s_0| < \eta$, $\|x - x_0\| < \eta$, $\|x\| \leq K_0$ and $\|x_0\| \leq K_0$, $\int_{\|y\| \leq K} |\tilde{K}\phi(s, x, y) - \tilde{K}\phi(s_0, x_0, y)| \leq \int_{\|y\| \leq K} \epsilon \|y\|^2 \mu(dy)$ for all $\mu \in \mathcal{M}_C$. Thus $\int_{\mathbb{R}^n - \{0\}} |\tilde{K}\phi(s, x, y) - \tilde{K}\phi(s_0, x_0, y)| \mu(dy)$ tends to 0 uniformly in $\mu \in \mathcal{M}_C$ when (s, x) tends (s_0, x_0) . On the other hand, making use one more time of the Taylor formula and of the continuity hypothesis (Definition 7.6 2.), it follows that $\int_{\mathbb{R}^n - \{0\}} \tilde{K}\phi(s_0, x_0, y) M(s, x, dy)$ is a continuous function of (s, x) . \square

Theorem 9.2 *The hypothesis are those of Theorem 8.3. Assume also that for K large enough, Γ_K is lower hemi continuous. Let h be continuous bounded from below. Let $v = \tilde{h}$ be a lower semi continuous function such that equations (103) and (104) are satisfied. In case of Levy diffusions v is a viscosity supersolution of*

$$\begin{cases} -\partial_u v(u, x) - f(u, x, Dv(u, x), D^2v(u, x), \tilde{K}v(u, x)) &= 0 \\ v(t, x) &= h(x) \end{cases} \quad (110)$$

at each point (t_0, x_0) such that $f(t_0, x_0, D\phi(t_0, x_0), D^2\phi(t_0, x_0), \tilde{K}\phi(t_0, x_0)) < \infty$ for all $\phi \in \mathcal{C}_b^{1,2}$ satisfying (107). Here $f(t, x, D\phi(t, x), D^2\phi(t, x), \tilde{K}\phi(t, x)) = \sup_{(a, b, \mu) \in \Gamma(t, x)} [(b^ D\phi(t, x) + \frac{1}{2} \text{Tr}(a D^2\phi(t, x)) + \int \tilde{K}\phi(t, x)(y) \mu(dy) - g(t, x, a, b, \mu)]$,*

and $\tilde{K}\phi(t, x)$ is given by equation (108)

In case of continuous diffusions, assume that the restriction of g to $\{(u, x, y), y \in \Gamma(u, x)\}$ is upper semi-continuous. Then v is a viscosity supersolution of

$$\left\{ \begin{array}{lcl} -\partial_u v(u, x) - f(u, x, Dv(u, x), D^2 v(u, x)) & = & 0 \\ v(t, x) & = & h(x) \end{array} \right. \quad (111)$$

at each point (t_0, x_0) such that $f(t_0, x_0, D\phi(t_0, x_0), D^2\phi(t_0, x_0)) < \infty$ for all $\phi \in C_b^{1,2}$ satisfying (107). In this case

$$f(t, x, z, \gamma) = \sup_{(a,b) \in \Gamma(t,x)} [(b^* z + \frac{1}{2} Tr(a\gamma) - g(t, x, a, b)] \quad (112)$$

Proof The proof follows the proof of Theorem 5.9 of [5], replacing Ito's formula by the martingale property. From the time consistency for the process $\Pi_{u,s}^{t_0, x_0}$, for all $\delta > 0$,

$$v(t_0, x_0) = \Pi_{t_0, t}^{t_0, x_0}(h(X_t)) = \Pi_{t_0, t_0+\delta}^{t_0, x_0}(\Pi_{t_0+\delta, t}^{t_0, x_0}(h(X_t))) \quad (113)$$

From the definition of v and the inequality $v \geq \phi$, it follows that

$$\Pi_{t_0+\delta, t}^{t_0, x_0}(h(X_t)) = v(t_0 + \delta, X_{t_0+\delta}) \geq \phi(t_0 + \delta, X_{t_0+\delta}) \quad (114)$$

From the definition of $\Pi_{t_0, t_0+\delta}^{t_0, x_0}$, it follows that for all $\theta \in \Theta(\Gamma)$,

$$v(t_0, x_0) \geq Q_{t_0, x_0}^\theta(v(t_0 + \delta, X_{t_0+\delta})) - \alpha_{t_0, t_0+\delta}(Q_{t_0, x_0}^\theta) \quad (115)$$

- We give details in case of diffusions with Levy generator. Let $\tilde{K}\phi(s, x)$ be given by equation (108). The function ϕ belongs to $C_b^{1,2}$. Thus it follows from the martingale property for Q_{t_0, x_0}^θ as stated in Theorem 1.1 of [32], from the definition of the penalty (Definition 7.7) and the equality $v(t_0, x_0) = \phi(t_0, x_0)$ that

$$\begin{aligned} 0 \geq & Q_{t_0, x_0}^\theta \left[\int_{t_0}^{t_0+\delta} \{ \partial_u \phi(u, X_u) + \frac{1}{2} Tr(a D^2 \phi)(u, X_u) + b^* D\phi(u, X_u) \} du \right. \\ & + Q_{t_0, x_0}^\theta \left[\int_{t_0}^{t_0+\delta} \left[\int_{\mathbb{R}^n - \{0\}} (\tilde{K}\phi(u, X_u)(y) M(u, X_u, dy)) \right] du \right] \\ & + Q_{t_0, x_0}^\theta \left[\left(\int_{t_0}^{t_0+\delta} -g(u, X_u, a(u, X_u), b(u, X_u), M(u, X_u)) du \right) \right] \end{aligned} \quad (116)$$

$f(t_0, x_0, D\phi(t_0, x_0), D^2\phi(t_0, x_0), \tilde{K}\phi(t_0, x_0)) = \sup_{(a,b,\mu) \in \Gamma(t_0, x_0)} [b^* D\phi(t_0, x_0) + \frac{1}{2} \text{Tr}(a D^2\phi(t_0, x_0)) + \int_{\mathbb{R}^n - \{0\}} \tilde{K}\phi(t_0, x_0)(y) \mu(dy) - g(t_0, x_0, a, b, \mu)]$. Let $(a_0, b_0, \mu_0) \in \Gamma$ such that

$$(b_0^* D\phi(t_0, x_0) + \frac{1}{2} \text{Tr}(a_0 D^2\phi(t_0, x_0)) + \int \tilde{K}\phi(t_0, x_0) d\mu_0 - g(t_0, x_0, a_0, b_0, \mu_0)) > f(t_0, x_0, D\phi(t_0, x_0), D^2\phi(t_0, x_0), \tilde{K}\phi(t_0, x_0)) - \epsilon \quad (117)$$

Let K such that $(a_0, b_0, \mu_0) \in \Gamma_K$ and Γ_K is lower hemicontinuous. There is then a continuous function $(s, x) \rightarrow a(s, x), b(s, x), M(s, x) = \theta^0(s, x) \in \Gamma_K$ such that $a(s_0, x_0), b(s_0, x_0), M(s_0, x_0) = (a_0, b_0, \mu_0)$. From the regularity properties of ϕ and from Lemma 9.1, one can write (116) as

$$0 \geq Q_{t_0, x_0}^\theta \left[\int_{t_0}^{t_0+\delta} (\xi(u, X_u) - g^{\theta^0}(u, X_u)) du \right] \quad (118)$$

where ξ and g^{θ^0} are continuous bounded.

- In case of continuous diffusions, it follows from equation (115) that

$$0 \geq Q_{t_0, x_0}^\theta \left[\int_{t_0}^{t_0+\delta} (\xi(u, X_u) - \bar{g}(u, X_u)) du \right] \quad (119)$$

where ξ is continuous bounded and \bar{g} is upper semi-continuous.

For all $\epsilon > 0$, there is $\eta > 0$, such that for $t_0 \leq u \leq t \leq t_0 + \eta$ and $\|x_0 - x\| < \eta$,

$$\xi(u, x) - \tilde{g}(u, x) \geq \xi(t_0, x_0) - \tilde{g}(t_0, x_0) - \epsilon \quad (120)$$

with \tilde{g} equal g^{θ^0} or \bar{g} . In case of Levy generators, let $K_0 = \|\xi\|_\infty + \|g^{\theta^0}\|_\infty$. In case of continuous diffusions, from hypothesis H_g , $|\bar{g}(u, x)| \leq 1 + \|x\|^m$. Let $K_0 = \|\xi\|_\infty + 1 + (Q_{t_0, x_0}^{\theta_0} (\sup_{t_0 \leq u \leq \eta} \|X_u\|^{2m}))^{\frac{1}{2}}$.

From Lemma 2.7 in case of Levy generators and from Proposition 2.3 of [5] in case of continuous diffusions, applied with the probability measure $Q_{t_0, x_0}^{\theta_0}$ there is $0 < \delta < \eta$ such that

$$Q_{t_0, x_0}^{\theta_0}(A) < \frac{\epsilon}{K_0} \text{ with } A = \{\omega \mid \sup_{t_0 \leq u \leq t_0+\delta} \|X_u - x_0\| > \eta\} \quad (121)$$

Dividing the inequality (116) by δ , making use of (117), (120), (121) and of Cauchy Schwarz inequality, it follows that

$$0 > f(t_0, x_0, D\phi(t_0, x_0), D^2\phi(t_0, x_0), \tilde{K}\phi(t_0, x_0)) - 3\epsilon$$

□

Remark 9.3 A different approach to viscosity solutions of fully non linear PDE is proposed in [31]. This approach is based on the existence of solutions to BSDE. In [31] the PDE is

$$-\partial_u v(u, x) - f(u, x, v(u, x), Dv(u, x), D^2 v(u, x))$$

with $f(t, x, y, z, \gamma) = \sup_a [\frac{1}{2} \text{Tr}(a\gamma) - g(t, x, y, z, a)]$, f is allowed to depend on y . However unlike for equation (112), the assumptions on function g (Assumption 5.1 of [31]) are quite restrictive: the domain of $g(t, x, \cdot)$ is independent of t and x , the function g is uniformly continuous in t , Lipschitz in z .

9.2 Viscosity subsolution

From now on we add a new hypothesis H' . In case of continuous diffusions we assume that the multivalued Borel mapping Γ has linear growth, i.e. There is a constant $K > 0$ such that $\forall(a, b) \in \Gamma(s, x)$, $\|a\|, \|b\| \leq K(1 + \|x\|)$. As in [5] Lemma 5.11, it follows from [21] II 5 Corollary 10 that for all $q \geq 1$, C , and $t > 0$, there is a constant K_1 such that for all y such that $\|y\| \leq C$ and all bounded $\gamma \in \Theta(\Gamma)_t^r$,

$$E_{Q_{r,y}^\gamma}(\sup_{s \leq u \leq t} (\|X_t - X_u\|^{2q}) \leq K_1(t-s)^q \quad (122)$$

and

$$\forall \epsilon > 0, \forall \eta > 0, \exists h > 0 \forall r > 0, \forall y, \|y\| \leq C, \forall \gamma \in \Theta(\Gamma)_t^r, Q_{r,y}^\gamma[\sup_{r \leq v \leq r+h} \|X_v - y\| > \eta] < \epsilon \quad (123)$$

Denote v^* the upper semi continuous envelope of v ,

$$v^*(s, x) = \limsup_{(s', x') \rightarrow (s, x)} v(s', x')$$

Let $\phi \in C_b^{1,2}$ such that

$$0 = v^*(t_0, x_0) - \phi(t_0, x_0) = \sup_{(s, x)} (v^*(s, x) - \phi(s, x)) \quad (124)$$

Theorem 9.4 Let h be continuous bounded from below. Assume that

1. In case of continuous diffusions, g satisfies hypothesis H_g and Γ is closed convex with linear growth.

2. In case of diffusions with Levy generator, $\Gamma = \Gamma_K$ for some $K > 0$.

Assume that f is upper semi continuous. Let $v = \tilde{h}$ be the lower semi continuous function as in Theorem 8.3. Then v^* is a viscosity subsolution of (1) in case of continuous diffusions and of (110) in case of diffusions with Levy generator

$\tilde{K}\phi(u, x)$ belongs to the vector space $\mathcal{C}(\mathbb{R}^n)$ of continuous functions on \mathbb{R}^n . Let \mathcal{M} be the vector space of measures generated by \mathcal{M}_C . The topology on E is $\sigma(E, \mathcal{M})$.

Proof It follows from the first part of the proof of Lemma 9.1 that $\tilde{K}\phi$ is continuous from $[0, T] \times \mathbb{R}^n$ to E . The functions $\phi_u, D\phi, D^2\phi$ are continuous and f is upper semi-continuous thus for all $n > 0$ there is $\eta_n > 0$ such that for $t_0 \leq u \leq t_0 + \eta_n$, and $\|x - x_0\| < \eta_n$, $\phi_u(u, x) + f(u, x, D\phi(u, x), D^2\phi(u, x), \tilde{K}\phi(u, x)) \leq \phi_u(t_0, x_0) + f(t_0, x_0, D\phi(t_0, x_0), D^2\phi(t_0, x_0), \tilde{K}\phi(t_0, x_0)) + \frac{1}{n}$. From equation (123) in case of continuous diffusions and inequality (11) in case of diffusions with Levy generator, there is $\alpha_n > 0$ such that for all $|u - t_0| \leq 1$ and $\|y - x_0\| \leq 1$ for all $\gamma \in \Theta(\Gamma)_t^u$,

$$Q_{u,y}^\gamma(A_n) < \frac{1}{n} \text{ with } A_n = \{\omega \mid \sup_{u \leq u' \leq u + \alpha_n} \|X_{u'} - y\| > \frac{\eta_n}{2}\}$$

Let $\delta_n = \inf(\eta_n, \alpha_n)$. For all $n > 0$ choose (t_n, x_n) such that $|t_n - t_0| < \frac{\delta_n}{2}$, $\|x_n - x_0\| < \frac{\eta_n}{2}$ and $\phi(t_n, x_n) - v(t_n, x_n) \leq \frac{\delta_n}{n}$. For all $n > 0$, there is a process $\gamma_n \in \Theta(\Gamma)_t^{t_n}$, such that

$$v(t_n, x_n) \leq E_{Q_{t_n, x_n}^{\gamma_n}}(h(X_t)) - \alpha_{t_n, t}(Q_{t_n, x_n}^{\gamma_n}) + \frac{1}{n}. \quad (125)$$

From the cocycle condition for the penalty associated to the probability measure $Q_{t_n, x_n}^{\gamma_n}$, the definition of $\Pi_{t_n + \delta_n, t}^{t_n, x_n}(h(X_t))$ and $\Pi_{t_n + \delta_n, t}^{t_n, x_n}(h(X_t)) = v(t_n + \delta_n, X_{t_n + \delta_n}) \leq \phi(t_n + \delta_n, X_{t_n + \delta_n})$

$$v(t_n, x_n) \leq E_{Q_{t_n, x_n}^{\gamma_n}}[\phi(t_n + \delta_n, X_{t_n + \delta_n}) - \int_{t_n}^{t_n + \delta_n} g(u, X_u, \mu_n(u, \omega)) du] + \frac{1}{n} \quad (126)$$

We apply the martingale property for $Q_{t_n, x_n}^{\gamma_n}$ and we make use of the definition of f . We can then finish the proof as the proof of the viscosity supersolution.

9.3 Viscosity solution and uniqueness

The following Theorem results from Theorems 9.2 and 9.4

Theorem 9.5 *Assume that all the previous hypothesis are satisfied. Assume furthermore that h is bounded from below and α Hölder-continuous for some $\alpha > 0$ in case of continuous diffusions and h is Lipschitz in case of diffusions with Levy generator. Let $v = \tilde{h}$ be a lower semi continuous function such that equations (103) and (104) are satisfied. Assume that the PDE (110) satisfies the comparison principle for functions bounded on compact spaces. Then the function v is continuous. It is the unique viscosity solution of (110).*

Proof As in [5] we prove first that v is continuous at (t, x) for all x . Making use of the equation (122) in case of continuous diffusions and of Lemma 2.7 for diffusions with levy generator, we prove as in [5] that $-v(s, x) - h(x)$ tends to 0 uniformly in x when s tends to t . The result follows then from Thm 9.2 and 9.4 and from Proposition 5.5 of [5]. For comparison results for non linear second order PDE we refer to [17, 14, 9].

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